Notes for An Introduction to Manifolds - Tu

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The following is a non-comprehensive list of side notes for Tu's *An Introduction to Manifolds*. The hope is that this PDF makes reading the text more smooth. Page notes appear here in the order that you will come across them on the page when reading the text. Diagrams will be included when necessary.

§1 Smooth Functions on a Euclidean Space

Page 6

- > ... Let
  > $$g_i(x) = \int_0^1 \frac{\partial f}{\partial x^i}(p + t(x-p))\,dt.$$
  > Then $g_i(x)$ is C^∞ ...

  Existence of the partial derivatives of $g_i(x)$ follows by Leibniz's rule for differentiation under the integral sign for functions of n variables (this says we can interchange the integral and a partial derivative operator). Continuity of the partial derivatives is proved directly and follows from the fact that the integrand is a uniformly continuous function of t.

§3 The Exterior Algebra of Multicovectors

Page 25

- > ... Check that the tensor product of multilinear functions is associative: if f, g, and h are multilinear functions on V, then
  > $$(f \otimes g) \otimes h = f \otimes (g \otimes h)$$

This follows immediately by the associativity of real-valued functions and the definition of the tensor product for multilinear functions.

Page 26

- > ... for every permutation $\sigma \in S_{k+\ell}$, there are $k!$ permutations τ in S_k that permute the first k arguments $v_{\sigma(1)}, \dots, v_{\sigma(k)}$ and leave the arguments of g alone; for all τ in S_k, the resulting permutations $\sigma\tau$ in $S_{k+\ell}$ contribute the same term to the sum, since
  >
  > $$\begin{aligned}(\operatorname{sgn}\sigma\tau) f(v_{\sigma\tau(1)}, \dots, v_{\sigma\tau(k)}) &= (\operatorname{sgn}\sigma\tau)(\operatorname{sgn}\tau) f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \\ &= (\operatorname{sgn}\sigma) f(v_{\sigma(1)}, \dots, v_{\sigma(k)}),\end{aligned}$$
  >
  > where the first equality follows from the fact that $(\tau(1), \dots, \tau(k))$ is a permutation of $(1, \dots, k)$ ...

  The reason the resulting permutations are $\sigma\tau$ is because when τ acts on $f \wedge g$ it only permutes the arguments of f, so then applying σ results in the arguments of f, $v_{\sigma(1)}, \dots, v_{\sigma(k)}$, being permuted. Now

  $$f(v_{\sigma\tau(1)}, \dots, v_{\sigma\tau(k)}) = (\operatorname{sgn}\tau) f(v_{\sigma(1)}, \dots, v_{\sigma(k)})$$

  because we can think of $f(v_{\sigma\tau(1)}, \dots, v_{\sigma\tau(k)})$ as taking $f(v_1, \dots, v_k)$ and applying τ and then σ (since σ is not a permutation of $(1, \dots, k)$, by applying σ we mean $\sigma f(v_1, \dots, v_k) = f(v_{\sigma(1)}, \dots, v_{\sigma(k)})$). Indeed, τ is a permutation of $(1, \dots, k)$, and f is alternating, so we may write $(\operatorname{sgn}\tau) f(v_1, \dots, v_k)$ instead. Then applying σ gives $(\operatorname{sgn}\tau) f(v_{\sigma(1)}, \dots, v_{\sigma(k)})$.

Page 27

- > ... We call a permutation $\sigma \in S_{k+\ell}$ a (k, ℓ)-shuffle if
  >
  > $$\sigma(1) < \cdots < \sigma(k) \qquad \text{and} \qquad \sigma(k+1) < \cdots < s(k+\ell))$$

  Every (k, ℓ)-shuffle can be formed by choosing an ordered list of k elements from the set $\{1, \dots, k+\ell\}$ and letting this ordered list be $\sigma(1) < \ldots < \sigma(k)$. Then let the remaining ℓ elements be $\sigma(k+1) < \cdots < \sigma(k+\ell)$. Hence there are also $\binom{k+\ell}{k}$ (k, ℓ)-shuffles in $S_{k+\ell}$.

Page 28

- $$f \wedge g = (\operatorname{sgn} \tau) g \wedge f$$

  To see why $\operatorname{sgn} \tau = (-1)^{k\ell}$ just notice that in the second row of the matrix

  $$\tau = \begin{bmatrix} 1 & \cdots & \ell & \ell+1 & \cdots & \ell+k \\ k+1 & \cdots & k+\ell & 1 & \cdots & k \end{bmatrix}$$

  the entry $k+i$, for $1 \leq i \leq \ell$, has k entries to the right of it which are all smaller, namely $1, \ldots, k$, so that the pairs $(k+i, j)$, for $1 \leq i \leq \ell$ and $1 \leq j \leq k$, are precisely all the inversions of τ and there are $k\ell$ of them.

Page 29

- ...For each $\mu \in S_{k+\ell}$ and each $\tau \in S_k$, there is a unique element $\sigma = \mu\tau^{-1} \in S_{k+\ell}$ such that $\mu = \sigma\tau$ ...

  In $\sigma = \mu\tau^{-1}$ and $\mu = \sigma\tau$ we regard τ as a permutation of $S_{k+\ell}$ by fixing $k+1, \ldots, k+\ell$.

Page 30

- ...A *homomorphism of graded algebras* is an algebra homomorphism that preserves the degree ...

  By "preserves the degree" we mean that if $A = \bigoplus_{k=0}^{\infty} A^k$ and $B = \bigoplus_{k=0}^{\infty} B^k$ are graded algebras and $\varphi : A \to B$ is a homomorphism of graded algebras, then φ takes A^k to B^k for all $k \geq 0$.

- $$A_*(V) = \bigoplus_{k=0}^{\infty} A_k(V) = \bigoplus_{k=0}^{n} A_k(V)$$

  The last equality follows from Corollary 3.31. However there is not harm in not assuming $A_k(V) = 0$ for all $k > n$ since we may verify $A_*(V)$ is an anticommutative graded algebra without this fact.

- > ...With the wedge product of multicovectors as multiplication, $A_*(V)$ becomes an anticommutative graded algebra ...

  From the definition of an algebra, this means in particular that the wedge product must distribute over sums. To check this directly is not difficult.

Page 31

- > ...A k-linear function on V is completely determined by its values on all k-tuples $(e_{i_1}, \ldots, e_{i_k})$. If f is alternating, then it is completely determined by its values on $(e_{i_1}, \ldots, e_{i_k})$ with $1 \leq i_i < \cdots < i_k \leq n$ ...

  To see this, write

  $$f(v_1, \ldots, v_k) = f\left(\sum v_1^i e_i, \ldots, \sum v_k^i e_i\right)$$

  where $v_j = \sum v_j^i e_i$. Then use the multilinearity of f. Now if f is alternating, and in $(e_{i_1}, \ldots, e_{i_k})$ we have $i_j = i_\ell$ for some $1 \leq j, \ell \leq k$ with $j \neq \ell$, then $f(e_{i_1}, \ldots, e_{i_k}) = -f(e_{i_1}, \ldots, e_{i_k})$ implying $f(e_{i_1}, \ldots, e_{i_k}) = 0$. Hence we only need to consider tuples $(e_{i_1}, \ldots, e_{i_k})$ with $i_1, \ldots, i_k$ distinct. Moreover, $(e_{i_1}, \ldots, e_{i_k})$ differs from the corresponding ascending order k-tuple by σ for some $\sigma in S_k$, so that f depends only on ascending order tuples $(e_{i_1}, \ldots, e_{i_k})$.

§4 Differential Forms on $\mathbb{R}^n$

Page 34

- > ...From any C^∞ function $f : U \to \mathbb{R}$, we can construct a 1-form df, called the differential of f, as follows. For $p \in U$ and $X_p \in T_pU$, define
  >
  > $$(df)_p(X_p) = X_p f$$

  There is a typo, we mean $T_p(\mathbb{R}^n)$ instead of T_pU. Moreover, this definition means $df : U \to T_p^*(\mathbb{R}^n)$ is the function sending p to the linear functional $(df)_p : T_p(\mathbb{R}^n) \to \mathbb{R}$ defined by $(df)_p(X_p) = X_p f$.

Page 35

- *Proof.* By definition,

$$(dx^i)_p\left(\left.\frac{\partial}{\partial x^j}\right|_p\right)=\left.\frac{\partial}{\partial x^j}\right|_p x^i=\delta^i_j.$$

□

To give more detail to the proof, after the identity above use Proposition 3.1 and the characterisation of α^i.

- ...To find a_j, apply both sides of (4.2) to the coordinate vector field $\partial/\partial x^j$:

$$df\left(\frac{\partial}{\partial x^j}\right)=\sum_i a_i\, dx^i\left(\frac{\partial}{\partial x^j}\right)=\sum_i a_i\delta^i_j=a_j.$$

On the other hand, by the definition of the differential,

$$df\left(\frac{\partial}{\partial x^j}\right)=\frac{\partial f}{\partial x^j}$$

The reason we may write

$$df\left(\frac{\partial}{\partial x^j}\right)=\sum_i a_i dx^i\left(\frac{\partial}{\partial x^j}\right)$$

is because $(dx^i)_p$ is linear (as it's a covector) for all $p\in U$ and $1\le i\le n$. Generally speaking, if we omit the $p\in U$, we may treat differentials df as linear maps sending operators to functions because for each $p\in U$, $(df)_p$ is a linear function. With this notion in mind,

$$df\left(\frac{\partial}{\partial x^j}\right)=\frac{\partial f}{\partial x^j}$$

means df is the linear map sending the differential operator $\partial/\partial x^j$ to the linear function $\partial f/\partial x^j$ on U.

Page 39

- > ... Since both sides of (4.3) are linear in ω and τ ...

  That is, if we let $D = d$, $a = \omega$, and $b = \tau$ in (4.3), and consider both sides independently (ignoring the equality which we have yet to prove).

- > ... moving the 1-form $(\partial g/\partial x^i)\, dx^i$ across the k-form dx^I ...

  Here we assume $\omega = f\, dx^I$ is a k-form.

Page 40

- > ... By (iii), $Df = df$ on C^∞ functions ...

  To see this, let $X = \sum \partial/\partial x^i$. Then (iii) implies that the coefficient functions of Df are precisely $\partial f/\partial x^i$ by writing $(Df)(X)$ in coordinates. This means exactly that $Df = df$ because the coefficient functions of df are also $\partial f/\partial x^i$.

Page 43

- > ... Proposition C expresses the fact that a 1-form on $\mathbb{R}^3$ is exact if and only if it is closed ...

  As noted at the bottom of page 43, this fact is a special case of the Poincaré lemma. It is not easy to prove even for 1-forms on $\mathbb{R}^3$, so the fact ”a 1-form on $\mathbb{R}^3$ is exact if and only if it is closed“ is not an obvious one.

§5 Manifolds

As a precursor to this section (and all other sections unless specified otherwise) subsets are given the subspace topology.

Page 49

- > ... p has a neighborhood U homeomorphic to an open ball $B := B(0, \epsilon) \subset \mathbb{R}^n$ with p mapping to 0 ...

  Indeed, we may assume U is homeomorphic to a ball B by noting that U is connected (so it must map to a union of connected balls) and then shrinking U if necessary so that its image is contained in a single ball. We may further assume p maps to 0 since every ball is homeomorphic.

Page 54

- > ... If $\{(U_\alpha, \phi_\alpha)\}$ is an atlas for M, then $\{(U_\alpha \cap V, \phi_\alpha|_{U_\alpha \cap V}\}$ is an atlas for V, where $\phi_\alpha|_{U_\alpha \cap V} : U_\alpha \cap V \to \mathbb{R}^n$ denotes the restriction of ϕ_α to the subset $U_\alpha \cap V$ ...

  There is a typo, we mean $\{(U_\alpha \cap V, \phi_\alpha|_{U_\alpha \cap V})\}$ instead of $\{(U_\alpha \cap V, \phi_\alpha|_{U_\alpha \cap V}\}$. We require V to be open so that $\phi_\alpha(U_\alpha \cap V)$ is an open subset of $\mathbb{R}^n$, and hence $(U_\alpha \cap V, \phi_\alpha|_{U_\alpha \cap V})$ is a chart.

- > ... In a manifold of dimension zero, every singleton subset is homeomorphic to $\mathbb{R}^0$ ...

  This is because about every point p in the manifold there is a chart (U, ϕ) such that $\phi : U \to \mathbb{R}^0$ is a homeomorphism (since the only nonempty open set in $\mathbb{R}^0$ is $\mathbb{R}^0$) implying $U = \{p\}$.

- > $$\mathrm{GL}(n, \mathbb{R}) := \{A \in \mathbb{R}^{n \times n} \mid \det A \neq 0\} = \det^{-1}(\mathbb{R} - \{0\})$$

  By $\det^{-1}(\mathbb{R} - \{0\})$ we mean the preimage under the determinant viewed as a function on matrices.

- > ...Since the determinant function
  >
  > $$\det\colon \mathbb{R}^{n\times n} \to \mathbb{R}$$
  >
  > is continuous, $\mathrm{GL}(n,\mathbb{R})$ is an open subset of $\mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$ and is therefore a manifold ...

  Recall that if $A \in \mathrm{GL}(n,\mathbb{R})$, then

  $$\det A = \sum_{\sigma\in S_n} \left(\operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} \right),$$

  where $\operatorname{sgn}(\sigma)$ is the sign of σ. This is a polynomial in the entries of A, so det is continuous. Now $\mathrm{GL}(n,\mathbb{R}) = \det^{-1}(\mathbb{R} - \{0\})$ means $\mathrm{GL}(n,\mathbb{R})$ is the preimage of an open set under a continuous function and therefore open. So $\mathrm{GL}(n,\mathbb{R})$ is a manifold by Example 5.12.

Page 56

> - *Proof.* Problem 5.5. □

The proof rests on the facts that in the product topology $U_\alpha \times V_i$ is open in $M \times N$ for open U_α and V_i in M and N respectively, and that if ϕ_α and ψ_i are homeomorphisms then so is $\phi_\alpha \times \psi_i$.

§6 Smooth Maps on a Manifold

As a precursor to this section, if M is a smooth manifold recall from page 53 that by a *chart* (U, ϕ) *about* p, we mean a chart in the differentiable structure of M such that $p \in U$. In general, an atlas or a chart on a smooth manifold means an atlas or a chart contained in the differentiable structure of the smooth manifold. This information can also be found in the first paragraph of section 6.2 but we state it here as it is useful to know for section 6.1 as well. Also, take note that if we compose smooth maps we need to verify the composites are defined on open subsets of Euclidean spaces so that the composition is defined on a, possible smaller, open set so we may talk about smoothness of the composition.

Page 62

- > ... Suppose (U, ϕ) and (V, ψ) are charts on N and M respectively such that $U \cap F^{-1}(V) \neq \varnothing$ ...

  Otherwise, The domain of $\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \mathbb{R}^m$ is empty and smoothness is vacuously true.

Page 66

- > ... matrix multiplication
  >
  > $$\mu : \mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$$
  >
  > is a C^∞ map ...

  This follows since smoothness of a map is equivalent to smoothness of all its component functions and the (i, j) component map is precisely $(AB)_{ij}$, a polynomial in the entries of A and B, a smooth function.

- > ... the (i, j)-minor of a matrix A is the determinant of the submatrix of A obtained by deleting the ith row and jth column ...

  Since the determinate is continuous, the map $A \to (j, i) - $ minor of A is continuous.

Page 68

- > ... When M and N are open subsets of Euclidean space and the charts are $(U, r^1, \ldots, r^n)$ and $(V, r^1, \ldots, r^m)$, the Jacobian matrix $[\partial F^i / \partial r^j]$, where $F^i = r^i \circ F$, is the usual Jacobian matrix from calculus ...

  As a note, whenever we write $[\partial F^i / \partial r^j]$ we always mean the usual Jacobian from calculus. That is, the relative charts are $(U, r^1, \ldots, r^n)$ and $(V, r^1, \ldots, r^m)$.

Page 69

- > ...Since ψ and ϕ are diffeomorphisms (Proposition 6.10), this last statement is equivalent to the local invertibility at p ...

  This is because any diffeomorphism is a local diffeomorphism and the composition of local diffeomorphisms is a local diffeomorphism.

§7 Quotients

Page 75

- > ...for every $(x, y) \in S \times S - R$, there is a basic open set $U \times V$ containing (x, y) such that $(U \times V) \cap R = \varnothing$ ...

  Since this is equivalent to $(S \times S) - R$ being open in $S \times S$ we don't need to specify that U and V are basic and the proof works just as well without assuming U and V are basic (however we may as well assume U and V are basic since a simple fact from point set topology says every topology has a basis and if $x \in U$ then there exists a basic open set about x contained in U).

Page 77

- > ...$\mathbb{R}P^1$ is homeomorphic to the quotient $S^1/\sim$, which is in turn homeomorphic to the closed upper semicircle with the two endpoints identified ...

  To see why the second homeomorphism is true, view S^1 as the set of complex numbers with modulus 1 and let $f : S^1 \to S^1$ be the map sending z to z^2. Then f is a continuous bijection mapping a compact space into a Hausdorff space and therefore a homeomorphism.

Page 79

- > ...R may be characterized as the set of matrices $[x\ y]$ in $S \times S$ of rank ≤ 1 ...

Actually, it's the set of matrices in $S \times S$ of rank precisely 1 because $[x\ y]$ is of rank 0 if and only if it's the zero matrix which is impossible because $S = \mathbb{R}^{n+1} - \{0\}$.

- > ... As the zero set of finitely many polynomials, R is a closed subset of $S \times S$ ...

  Indeed, if $f_1, \ldots, f_{\binom{n+1}{2}} : S \times S \to \mathbb{R}$ are the maps taking $[x\ y]$ to each of its (i,j)-minors, then

  $$R = \bigcap_{i=1}^{\binom{n+1}{2}} f_i^{-1}(\{0\})$$

  which is the intersection of finitely many closed sets (because each f_i is continuous and $\{0\}$ is closed in $\mathbb{R}$) hence closed. The "finitely many" here doesn't matter of course because the intersection of arbitrarily many closed sets is closed.

Page 80

- > $$U_i := \{[a^0, \ldots, a^n] \in \mathbb{R}P^n \mid a^i \neq 0\}$$

  Let $\pi : \mathbb{R}^{n+1} - \{0\} \to \mathbb{R}P^n$ be the projection map. Each U_i is open. Indeed, $\pi^{-1}(U_i) = \{(a^0, \ldots, a^n) \in \mathbb{R}^{n+1} - \{0\} \mid a^i \neq 0\}$ and for each $(a^0, \ldots, a^n) \in \pi^{-1}(U_i)$ there exists an open set V_i about a_i not containing zero (since $a_i \neq 0$), so choosing open sets V_j about a_j for $j \neq i$ we have that $V_1 \times \cdots \times V_n$ is an open set about $(a^0, \ldots, a^n)$ contained in $\pi^{-1}(U_i)$.

Page 81

- > $$(\phi_1 \circ \phi_0^{-1})(x) = \left(\frac{1}{x^1}, \frac{x^2}{x^1}, \frac{x^3}{x^1}, \ldots, \frac{x^n}{x^1}\right)$$

  Here we abuse notation a little bit by writing $x = (x^1, \ldots, x^n)$ for a point in $\phi_0(U_0 \cap U_1)$ as well as the coordinate functions on U_0.

§8 The Tangent Space

Page 87

- ...It is easily checked that $\partial/\partial x^i|_p$ satisfies the derivation property ...

  The computation is as follows:

$$\begin{aligned} \left.\frac{\partial}{\partial x^i}\right|_p (fg) &= \left.\frac{\partial}{\partial r^i}\right|_{\phi(p)} ((fg)\circ\phi^{-1}) \\ &= \left.\frac{\partial}{\partial r^i}\right|_{\phi(p)} ((f\circ\phi^{-1})(g\circ\phi^{-1})) \\ &= \left(\left.\frac{\partial}{\partial r^i}\right|_{\phi(p)} (f\circ\phi^{-1})\right) g(p) + \left(\left.\frac{\partial}{\partial r^i}\right|_{\phi(p)} (g\circ\phi^{-1})\right) f(p) \\ &= \left(\left.\frac{\partial}{\partial x^i}\right|_p f\right) g(p) + \left(\left.\frac{\partial}{\partial x^i}\right|_p g\right) f(p) \end{aligned}$$

- ...Since (8.1) is independent of the representative of the germ ...

  We show why (8.1) is independent of the representative of the germ and give some detail to why it is a derivation. Indeed, if f and g are two representatives for the same germ then they agree on some open set U about $F(p)$. Then $f\circ F$ and $g\circ F$ agree on the open set $F^{-1}(U)$ (recall F is C^∞ hence continuous). Moreover, $F_*(X_p)$ is immediately seen to be a derivation because X_p is a derivation and composition of functions distributes over products of functions. Also, F_* is linear by definition.

Page 88

- $$T_pN \xrightarrow{F_{*,p}} T_{F(p)}M \xrightarrow{G_{*,F(p)}} T_{G(F(p))}P$$

  Here the added subscripts p and $F(p)$ on $F_{*,p}$ and $G_{*,F(p)}$ respectively are to help indicate $F_{*,p}$ is the induced map on T_pN and $G_{*,F(p)}$ is the induced map on $T_{F(p)}M$. We really mean F_* and G_*

- $$((G_* \circ F_*)X_p)f = (G_*(F_*X_p))f = (F_*X_p)(f \circ G) = X_p(f \circ G \circ G)$$

  The order of equalities here is a little strange. By definition, $(G_*(F_*X_p))f = (F_*X_p)(f \circ G)$ and by definition again $(F_*X_p)(f \circ G) = X_p(f \circ G \circ G)$. Therefore we may conclude $((G_* \circ F_*)X_p)f = (G_*(F_*X_p))f$. This last equality precisely means $(G \circ F)_{*,p} = G_{*,F(p)} \circ F_{*,p}$.

Page 90

- ... An isomorphism of vector spaces carries a basis to a basis ...

  While this is true, technically the fact we are using is that if the image of a set of vectors is a basis then using the inverse isomorphism sends this basis back to our original set of vectors and therefore the original set of vectors must have also been a basis since an isomorphism of vector spaces carries a basis to a basis.

Page 91

- ... a C^∞ map $F : N \to M$ between two manifolds of the same dimension is locally invertible at a point $p \in N$ if and only if its differential $F_{*,p} : T_pN \to T_{F(p)}M$ at p is an isomorphism ...

  By the inverse function theorem for manifolds (Theorem 6.26), F is locally invertible at p if and only if the determinate of the Jacobian matrix $[\partial F^i/\partial x^j(p)]$ is nonzero, and since this matrix represents the linear map $F_{*,p}$ by Proposition 8.11, this condition is equivalent to $F_{*,p}$ being an isomorphism. This remark is especially elegant since it reduces locally invertibility of F at p to checking if $F_{*,p}$ is either injective or surjective because $F_{*,p}$ is a linear map between vector spaces of the same dimension.

Page 92

- . . . Alternative notations for $c'(t_0)$ are

$$\frac{dc}{dt}(t_0) \quad \text{and} \quad \left.\frac{d}{dt}\right|_{t_0} c$$

Another notation used in the text is

$$\left.\frac{d}{dt}\right|_{t_0} c(t).$$

Page 93

- . . . relative to the basis $\{\partial/\partial x^i|_p\}$ for $T_{c(t)M}$ . . .

This is a typo. We mean $\{\partial/\partial x^i|_{c(t)}\}$.

- *Proof.* Problem 8.5. □

The proof proceeds as in Exercise 8.14. By the definition of $c'(t)$ we know

$$c'(t) = \sum_{i=1}^{n} a^i(t) \left.\frac{\partial}{\partial x^i}\right|_{c(t)}.$$

Then we evaluate both sides on x^i to find $a^i(t) = \dot{c}^i(t)$.

Page 94

- . . . Let $(U, \phi) = (U, x^1, \ldots, x^n)$ be a chart centered at p . . .

We have not needed this fact until now, but given a manifold M and a point $p \in M$ we may assume there exists a chart centered about p. By abuse of notation, if (U, ϕ) is a chart about p then there exists a ball B_p, centered at p, such that $p \in B \subseteq U$. Then $(B, \phi|_B)$ is a chart about p and since all balls are homeomorphic we may assume $\phi|_B(p) = \mathbf{0}$.

- > ... by Proposition 8.8,
  >
  > $$c'(0) = (\phi^{-1})_*\alpha_*\left(\frac{d}{dt}\bigg|_{t=0}\right) = (\phi^{-1})_*\left(\sum a^i\frac{\partial}{\partial r^i}\bigg|_{\mathbf{0}}\right) = \sum a^i\frac{\partial}{\partial x^i}\bigg|_p = X_p$$

We're really using the fact $\phi : U \to \mathbb{R}^n$ is a diffeomorphism onto its image so $\phi^{-1} : \phi(U) \to U$ is a diffeomorphism, and hence by Corollary 8.6 and Proposition 8.8 we may conclude

$$(\phi^{-1})_*\left(\frac{\partial}{\partial r^i}\bigg|_{\mathbf{0}}\right) = \frac{\partial}{\partial x^i}\bigg|_p.$$

Page 95

- > ... $d/dt|_{t=0}gc(t) = gc'(0)$ by $\mathbb{R}$-linearity and Proposition 8.15 ...

We give a little more detail. Let g_{ij} be the (i,j)-entry of g and for each t let $c(t)_{ij}$ be the (i,j)-entry of $c(t)$. Then by matrix multiplication we have

$$gc(t)_{ij} = \sum_{k=1}^{n} g_{ik}c(t)_{kj}.$$

Viewing the matrices in $\mathbb{R}^{n\times n}$ we may differentiate entrywise so that

$$\frac{d}{dt}gc(t)_{ij} = \frac{d}{dt}\sum_{k=1}^{n} g_{ik}c(t)_{kj} = \sum_{k=1}^{n} g_{ik}\frac{d}{dt}c(t)_{kj},$$

where the last equality holds by $\mathbb{R}$-linearity since the entries of g are constants. Identifying the tangent space $T_{c(t)}(\mathrm{GL}(n,\mathbb{R}))$ with $\mathbb{R}^{n\times n}$ the basis $\{\partial/\partial x^i|_{c(t)}\}$ becomes the standard euclidean basis $\{e_i\}$ so Proposition 8.15 tells us that the $n \times n$ entries of the column vector $c'(t)$ are precisely $\dot{c}^i(t) = d/dt(c(t)_{kj})$ (we may need to reorder some of the entries but this is permitted because vector spaces are isomorphic up to reordering). Hence

$$\frac{d}{dt}gc(t)_{ij} = \frac{d}{dt}\sum_{k=1}^{n} g_{ik}c'(t)_{kj} = gc'(t)_{ij}.$$

Evaluating the derivative at 0 then gives $d/dt|_{t=0}gc(t) = gc'(0)$ because this formula holds entrywise.

Page 96

- > . . . If U is an open subset of a manifold M, then the inclusion $\iota : U \to M$ is both an immersion and a submersion . . .

  This is because $i : U \to M$ is a diffeomorphism onto its image implying the differential $i_{*,p} : T_pU \to T_pM$ for every $p \in U$ is an isomorphism of vector spaces.

Page 97

- > . . . Critical points and critical values of the function $f(x, y, z) = z$ . . .

  The reason why the critical points are as indicated in Figure 8.4 is because the tangent space (viewed geometrically) at these points lie in a fixed z-plane and the differential map takes the tangent vectors in these planes to the vectors in $\mathbb{R}$ emanating from the corresponding critical value and terminating at their z-coordinate (which is the critical value since $f(x, y, z) = z$). But for each tangent space the z-coordinate is fixed so all of these tangent vectors are sent to the zero vector emanating at the corresponding critical value.

§9 Submanifolds

Page 100

- > . . . $U \cap S$ is defined by the vanishing of $n - k$ of the coordinate functions. By renumbering the coordinates, we may assume that these $n - k$ coordinate functions are $x^{k+1}, \ldots, x^n$ . . .

  Another way of saying this is that $U \cap S$ is precisely the common zero set of $n-k$ of the coordinate functions. That is $U \cap S = (x^{k+1})^{-1}(\{0\}) \cap \ldots \cap (x^n)^{-1}(\{0\})$.

- > . . . Note that $(U \cap S, \phi_S)$ is a chart for S in the subspace topology . . .

  Indeed, give S the subspace topology from N. Then $U \cap S$ is by definition an open set in S and $p \in U \cap S$. So $U \cap S$ is a coordinate neighborhood of

p. Since the restriction of a homeomorphism is a homeomorphism under the subspace topology and $\phi(U \cap S)$ can be identified with a subspace of $\mathbb{R}^k$ (since $n-k$ of the coordinate functions vanish) we conclude that $\phi_S : U \cap S \to \mathbb{R}^k$ is a homeomorphism onto its image and therefore a coordinate map. Hence the pair $(U \cap S, \phi_S)$ is a chart on S.

Page 101

- > ...there is no adapted chart containing p, since any sufficiently small neighborhood U of p in $\mathbb{R}^2$ intersects S in infinitely many components ...

  The idea here is that for any U there are points in $U \cap \Gamma$ with nonzero x or nonzero y coordinate so that $U \cap \Gamma$ is not defined by the vanishing of either coordinate function.

Page 102

- > ...Therefore, $(\psi_S \circ \phi_S^{-1})(x^1, \ldots, x^k) = (y^1, \ldots, y^k)$. Since $y^1, \ldots, y^k$ are C^∞ functions of $x^1, \ldots, x^k$ (because $\psi \circ \phi^{-1}(x^1, \ldots, x^k, 0, \ldots, 0)$ is C^∞), the transition function $\psi_S \circ \phi_S^{-1}$ is C^∞ ...

  Here we explain the argument in a different manner. Each element of $\phi_S(U \cap V \cap S)$ may be written as $(x^1(p), \ldots, x^k(p))$ for some $p \in U \cap V \cap S$. Hence $\psi_S \circ \phi_S^{-1}$ takes $(x^1(p), \ldots, x^k(p))$ to $(y^1(p), \ldots, y^k(p))$. This is what we mean when we write $(\psi_S \circ \phi_S^{-1})(x^1, \ldots, x^k) = (y^1, \ldots, y^k)$. Now this function is C^∞ if and only if all its component functions are C^∞ and this is true because all the component functions of

  $$\psi \circ \phi^{-1}(x^1, \ldots, x^k, 0, \ldots, 0) = (y^1, \ldots, y^k, 0, \ldots, 0)$$

  are C^∞ since $\psi \circ \phi^{-1}$ is C^∞.

Page 104

- > ...the differential $f_{*,p}$ equals $g_{*,p}$ at every point $p \in N$ ...

What we mean by equality here is that $T_{g(p)}\mathbb{R}$ and $T_{f(p)}\mathbb{R}$ can be identified and $g_{*,p} = f_{*,p}$ after this identification. Indeed, we identify these tangent spaces via the map $X_{g(p)}(h) \mapsto X_{g(p)}(c{\cdot}h)$ where h is a germ at $g(p)$ and $(c{\cdot}h)(x) = h(x{+}c)$. So the map acts on the input of h. Notice that this induces an identification of germ spaces $C^\infty_{g(p)}\mathbb{R}$ and $C^\infty_{f(p)}\mathbb{R}$ where the map is given by $h \mapsto c \cdot h$. The fact this map is well-defined is seen as follows. If h and h' are in the same class then they agree on some neighborhood U of $g(p)$, and setting $V = U - c$, V is a neighborhood of $f(p)$. So, if $y \in V$ then $y = x - c$ for some $x \in U$ and

$$(c \cdot h)(y) = h(y + c) = h(x) = h'(x) = h'(y + c) = (c \cdot h')(y).$$

Now to show $g_{*,p} = f_{*,p}$, let $X_p \in T_pN$. Then

$$\begin{aligned} g_{*,p}(X_p)h &= X_p(h \circ g) \\ &\mapsto X_p(h \circ (g + c)) \\ &= X_p(h \circ (f + c)) \\ &= X_p((c \cdot h) \circ f) \\ &= f_{*,p}(X_p)(c \cdot h), \end{aligned}$$

as desired. Notice that when we write $X_p(h \circ g) \mapsto X_p(h \circ (g + c))$ we are considering $X_p(h \circ g)$ as a derivation in $T_{g(p)}\mathbb{R}$ so that the identification map $X_{g(p)}(h) \mapsto X_{g(p)}(c \cdot h)$ acts on the input of h and not the input of $h \circ g$.

Page 105

- > ...the $m \times n$ Jacobian matrix $[\partial F^i/\partial x^j(p)]$ has rank m. By renumbering the F^i and x^j's, we may assume that the first $m \times m$ block $[\partial F^i/\partial x^j(p)]_{1\le i,j\le m}$ is nonsingular ...

Indeed, if an $m \times n$ matrix A has rank r then this implies the existence of a nonsingular $r \times r$ submatrix of A and all larger submatrices are singular. The proof proceeds as follows. Since the rank of A is r, there are r linearly independent columns in A. Let B be the $m \times r$ submatrix of A formed by these columns. Then the rank of B is r as well and since the column rank is equal to the row rank for any matrix, there are r linearly independent rows. Letting C be the $r \times r$ submatrix of B by using those rows, C is a $r \times r$ nonsingular submatrix of A. If there was a larger nonsingular submatrix D, then the columns of A that include the columns of D must be independent. This means that the rank of A is larger than r, which is impossible.

Page 106

- > **Lemma 9.10.** Let $F : N \to \mathbb{R}^m$ be a C^∞ map on a manifold N of dimension n and let S be the level set $F^{-1}(\mathbf{0})$. If relative to some coordinate chart $(U, x^1, \ldots, x^n)$ about $p \in S$, the Jacobian determinant $\partial(F^1, \ldots, F^m)/\partial(x^{j_1}, \ldots, x^{j_m})(p)$ is nonzero, then in some neighborhood of p one may replace $x^{j_1}, \ldots, x^{j_m}$ by $F_1, \ldots, F^m$ to obtain an adapted chart for N relative to S.

  There is no independent proof of this theorem because the proof is literally the second paragraph of the proof of the regular level set theorem.

§10 Categories and Functors

Page 112

> - *Proof.* Problem 10.2. □

The proof proceeds as follows. If $f : A \to B$ is an isomorphism, then there exists a map $g : B \to A$ such that $f \circ g = \mathrm{id}_A$ and $g \circ f = \mathrm{id}_B$. Then by the definition of a covariant functor

$$\mathcal{F}(g) \circ \mathcal{F}(f) = \mathcal{F}(g \circ f) = \mathcal{F}(\mathrm{id}_A) = \mathrm{id}_{\mathcal{F}(A)},$$

and by a symmetric argument $\mathcal{F}(f) \circ \mathcal{F}(g) = \mathrm{id}_{\mathcal{F}(B)}$. Therefore $\mathcal{F}(f) : \mathcal{F}(A) \to \mathcal{F}(B)$ is an isomorphism.

Page 113

> - *Proof.* Problem 10.3. □

Part (i) of the proof is obvious by the definition of the dual map. For (ii) let $\alpha : W \to \mathbb{R}$ be a linear functional on W. Then $(g \circ f)^\vee(\alpha) = \alpha \circ g \circ f$ and $(f^\vee \circ g^\vee)(\alpha) = f^\vee(\alpha \circ g) = \alpha \circ g \circ f$.

- > ...if $f : V \to W$ is an isomorphism, then so is its dual $f^\vee : W^\vee \to V^\vee$ (cf. Proposition 10.3) ...

The proof that a contravariant functor takes isomorphism to isomorphisms (reversing the arrow) is literally the exact same proof as given in the note regarding Proposition 10.3 just with the alternate condition

$$\mathcal{F}(g) \circ \mathcal{F}(f) = \mathcal{F}(f \circ g) = \mathcal{F}(\mathrm{id}_B) = \mathrm{id}_{\mathcal{F}(B)}.$$

Page 114

- *Proof.* Problem 10.6. □

Again, (i) is obvious. Statement (ii) is proved as follows. Let $f \in A_k(W)$ and $u_1, \ldots, u_k \in U$. Then

$$\begin{aligned}((L \circ K)^* f)(u_1, \ldots, u_k) &= f((L \circ K)(u_1), \ldots, (L \circ K)(u_k)) \\ &= f(L(K(u_1)), \ldots, L(K(u_k))) \\ &= (L^* f)(K(u_1), \ldots, K(u_k)) \\ &= ((K^* \circ L^*) f)(u_1, \ldots, u_k).\end{aligned}$$

§11 The Rank of a Smooth Map

Page 118

- $$\begin{aligned} f_{*,p} \text{ is injective} &\iff n \leq m \text{ and } \mathrm{rk}[\partial f^i/\partial x^j(p)] = n, \\ f_{*,p} \text{ is surjective} &\iff n \geq m \text{ and } \mathrm{rk}[\partial f^i/\partial x^j(p)] = m. \end{aligned} \tag{11.4}$$

The forward implications are obvious since the rank of a matrix is the dimension of the image space. For the reverse implications, the conditions $n \leq m$ and $n \geq m$ are necessary for $f : N \to M$ to be an injection respectively surjection. Now suppose $n \leq m$ and $\mathrm{rk}[\partial f^i/\partial x^j(p)] = n$. If we have a vector v such that $f_{*,p}(v) = 0$, then $f_{*,p}(v)$ is a linear combination of the columns of $[\partial f^i/\partial x^j(p)]$ which are all linearly independent and so the coefficients of this linear combination must be all 0. Hence $v = 0$, and so $f_{*,p}$ is injective. If $n \geq m$ and $\mathrm{rk}[\partial f^i/\partial x^j(p)] = m$, then the dimension of the the image space of $f_{*,p}$ is m and hence all of M. This precisely means $f_{*,p}$ is surjective.

Page 119

- > …By Proposition 11.4, the following theorems are simply special cases of the constant rank theorem …

  For the immersion and submersion theorems we make use of the following fact. For N and M manifolds of dimensions n and m respectively, if a smooth map $f : N \to M$ is an immersion $n \leq m$ and if it is a submersion then $n \geq m$.

Page 122

- > …These authors define a submanifold to be the image of any one-to-one immersion with the topology and differential structure inherited from f …

  We explain what we mean by "differential structure inherited from f". Suppose $f : N \to M$ is a smooth map of manifolds and is a one-to-one immersion. We want to define a differential structure on $f(N)$. To do this, suppose (U, ϕ) is a chart about a point $p \in N$. Then $(f(U), \phi \circ f^{-1})$ is a chart about $f(p) \in f(N)$. Take the collection of all such charts about all points $p \in N$. This collection covers $f(N)$ because f is bijective. We need to check they are pairwise C^∞-compatible. If $(f(U), \phi \circ f^{-1})$ and $(f(V), \psi \circ f^{-1})$ are two charts about $f(p)$ then $\phi \circ f^{-1} \circ f \circ \psi^{-1} = \phi \circ \psi^{-1}$ and $\psi \circ f^{-1} \circ f \circ \phi^{-1} = \psi \circ \phi^{-1}$ which are both smooth since N is assumed a manifold. The maximal atlas of such charts is then the smooth structure on $f(N)$ inherited from f.

Page 123

- > …By the immersion theorem (Theorem 11.5), there are local coordinates $(U, x^1, \ldots, x^n)$ near p and $(V, y^1, \ldots, y^m)$ near $f(p)$ such that $f : U \to V$ has the form
  > $$(x^1, \ldots, x^n) \mapsto (x^1, \ldots, x^n, 0, \ldots, 0).$$

  This is a typo. We mean

  $$(x^1, \ldots, x^n) \mapsto (y^1, \ldots, y^n, 0, \ldots, 0).$$

We may view the statement in this form instead of

$$(\psi \circ f \circ \phi^{-1})(r^1, \dots, r^n) = (r^1, \dots, r^n, 0, \dots, 0)$$

by precomposing with ϕ (since $x^i = r^i \circ \phi$) and noticing that we may write $(r^1, \dots, r^n, 0, \dots, 0)$ as $(y^1, \dots, y^n, 0, \dots, 0)$ given we compose with ψ^{-1} (since $y^i = r^i \circ \psi$ and so $r^i = y^i \circ \psi$).

Page 124

- > ...the inclusion i is given by
  >
  > $$(y^1, \dots, y^n) \mapsto (y^1, \dots, y^n, 0, \dots, 0),$$
  >
  > which shows that i is an immersion ...

  Indeed, let n and m be the dimensions of N and M respectively. Then $n \leq m$ and $\mathrm{rk}[\partial y^i / \partial y^j(p)] = n$. So the inclusion is an immersion by Equation 11.4.

Page 125

- > ...$\mathrm{SL}(n, \mathbb{R}) \times \mathrm{SL}(n, \mathbb{R})$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R})$ ...

  This follows from Problem 9.8 and Example 9.13.

Page 126

- > ...the tangent plane to N at p is defined by the equation
  >
  > $$\sum_{i=1}^{3} \frac{\partial f}{\partial x^i}(p)(x^i - p) = 0. \qquad (11.5)$$

  Indeed, a plane is defined by a normal vector and a point p on the plane. We know $p \in T_pN$ (this is just the zero vector) and we've shown any tangent vector $v = \langle v^1, v^2, v^3 \rangle$ is perpendicular to $\langle \partial f / \partial x^1(p), \partial f / \partial x^2(p), \partial f \partial x^3(p) \rangle$ since

  $$\sum_{i=1}^{3} \frac{\partial f}{\partial x^i}(p) v^i = 0.$$

Hence the equation above must be an equation defining the tangent plane T_pN.

§12 The Tangent Bundle

Page 131

- > ...the relative topology on TV as a subset of TU is the same as the topology induced from the bijection $\tilde{\phi}|_{TV} : TV \to \phi(V) \times \mathbb{R}^n$ ...

  Indeed, A is open in TV (as a subspace of TU) if and only if $A = TV \cap B$ for some open set B in TU. Now B is open in TU if and only if $\tilde{\phi}(B)$ is open in $\tilde{\phi}(U) \times \mathbb{R}^n$. But $\tilde{\phi}(A) = \tilde{\phi}(B) \cap (\tilde{\phi}(V) \times \mathbb{R}^n)$ because $\tilde{\phi}|_{TV}$ is a bijection. So $\tilde{\phi}(B)$ is open in $\tilde{\phi}(U) \times \mathbb{R}^n$ if and only if $\tilde{\phi}(A)$ is open in $\tilde{\phi}(V) \times \mathbb{R}^n$ which happens if and only if A is open in TV (as a topological space induced from the bijection $\tilde{\phi}|_{TV}$). Hence the topologies are the same.

Page 132

> - *Proof.* Problem 12.1. □

We prove the tangent bundle TM of a manifold M is Hausdorff. Let $v, u \in TM$. If $v \in T_pM$ and $u \in T_qM$ for $p \neq q$, then we choose disjoint open sets U_p and U_q about p and q respectively because M is Hausdorff. We may assume they are coordinate neighborhoods by intersecting them with coordinate neighborhoods if necessary. Then $T(U_p)$ and $T(U_q)$ are open sets about v and u respectively and disjoint by construction. If $p = q$, choose a coordinate open neighborhood U_p about p. Then $v, u \in T(U_p)$ and $T(U_p)$ is homeomorphic to an open subset of $\mathbb{R}^{2n}$ so it must be Hausdorff.

Page 135

- > ... A smooth map $f : N \to M$ of manifolds induced a bundle map $(f, \tilde{f})$, where $\tilde{f} : TN \to TM$ is given by
  >
  > $$\tilde{f}(p, v) = (f(p), f_*(v)) \in \{f(p)\} \times T_{f(p)}M \subset TM$$
  >
  > for all $v \in T_pN$ ...

  We verify $(f, \tilde{f})$ is a bundle map. Indeed, $(\pi_F \circ \tilde{f})(p, v) = \pi_F(f(p), f_*(v)) = f(p)$ and $(f \circ \pi_E)(p, v) = f(p)$ so the associated diagram is commutative. Moreover $\tilde{f}$ is linear on each fiber because $\tilde{f}|_{T_pN} = f_{*,p}$ and we know the differential is linear.

- > ... In this category it makes sense to speak of an isomorphism of vector bundles *over* M ...

  For the sake of completeness, a bundle map $\overline{f} : E \to F$ over M is a bundle isomorphism over M if there is a bundle map $\bar{g} : F \to E$ over M such that $\bar{g} \circ \overline{f} = \mathbb{1}_E$ and $\overline{f} \circ \bar{g} = \mathbb{1}_F$.

Page 137

- > ... This proves that $s + t$ is a C^∞ map on V and hence at p ...

  Indeed, we've shown that $(\phi \circ (s + t))$ is smooth and so composing with the diffeomorphism ϕ^{-1} shows $s + t$ is smooth.

- > ... We omit the proof, since it is similar to that of (i) ...

  We include the proof here. As before, it's clear $fs(p)$ is a section of E. To show it's C^∞, fix $p \in M$, let V be a trivializing open set for E containing p with C^∞ trivialization

  $$\phi : \pi^{-1}(V) \to V \times \mathbb{R}^r,$$

and suppose

$$(\phi \circ s)(q) = (q, a^1(q), \ldots, a^r(q))$$

for $q \in V$. Then the a^i are smooth maps as before. Since ϕ is linear on fibers,

$$(\phi \circ fs)(q) = (q, f(q)a^1(q), \ldots, f(q)a^r(q)).$$

Composing with ϕ^{-1} then shows fs is a C^∞ section of E.

- > ...For any open set $U \subset M$, one can also consider the vector space $\Gamma(U, E)$ of C^∞ sections of E over U ...

  For completeness, a section of E over U is a map $s : U \to E$ such that $\pi \circ s = \mathbb{1}_U$. A smooth section of E over U also demands s be a smooth map.

Page 138

- > ...ϕ^{-1} then carries the C^∞ frame $\bar{e}_1, \ldots, \bar{e}_r$ of the product bundle $U \times \mathbb{R}^r$ to a C^∞ frame $t_1, \ldots, t_r$ for E over U ...

  Indeed, ϕ^{-1} is an isomorphism on each fiber and since isomorphisms take a basis to a basis ϕ^{-1} carries $\bar{e}_1, \ldots, \bar{e}_r$ to $t_1, \ldots, t_r$. The frame $t_1, \ldots, t_r$ is C^∞ since ϕ^{-1} is smooth.

- > ...Fix a point $p \in U$ and choose a trivializing open set $V \subset U$ for E containing p ...

  We can assume $V \subset U$ by choosing any trivializing open set V containing p and intersecting it with U if necessary.

§13 Bump Functions and Partitions of Unity

Page 143

- > $$\sigma(x) = \rho(||x||) = 1 - g\left(\frac{||x||r^2 - a^2}{b^2 - a^2}\right). \qquad (13.2)$$

  There is a typo. We mean $||x||$ instead of $||x||r^2$.

Page 144

- > ...If $q \notin U$, then $q \notin \operatorname{supp} \rho$, and so there is an open set containing q on which $\tilde{f}$ is 0, since $\operatorname{supp} \rho$ is closed. Therefore $\tilde{f}$ is also C^∞ at every point $q \notin U$ ...

  Indeed, the above says that for every $q \notin U$ there exists an open set about q such that $\tilde{f}$ is 0. Clearly 0 is a smooth function on V so $\tilde{f}$ is smooth at $q \notin U$ by definition.

Page 146

> - *Proof.* Problem 13.1. □

We prove this statement. If $q \in \operatorname{supp}(\sum \rho_i)$ then $(\sum \rho_i)(q) \neq 0$ so $q \in \operatorname{supp} \rho_i$ for some i and therefore $q \in \bigcup \operatorname{supp} \rho_i$.

- > ...$\{\varphi_i\}$ is a partition of unity ...

  Indeed, it is locally finite since for any $q \in M$, q lies in finitely many (maybe all) of the sets $W_{q_1}, \ldots, W_{q_m}$, the support of ψ_{q_i} lies in W_{q_i}, and $\varphi_i = \psi_{q_i}/\psi$. Moreover this partition is smooth since $\varphi_i = \psi_{q_i}/\psi$ and $\psi > 0$.

Page 147

- > ...$\{\rho_\alpha\}$ is a partition of unity subordinate to $\{U_\alpha\}$ ...

  Indeed, we only elaborate on the smoothness of ρ_α and that $\{\rho_\alpha\}$ is locally finite. Since ρ_α is a sum of smooth functions it is smooth. Also, $\{\rho_\alpha\}$ is locally finite since $\{\varphi_i\}$ is locally finite. Indeed, if $q \in M$ then q lies in finitely many of the supports of the φ_i and therefore lies in finitely many of the supports of the ρ_α since we group the collection $\{\phi_i\}$ according to $\tau(i)$.

§14 Vector Fields

Page 150

- > ...This lemma is a special case of Proposition 12.12, with E the tangent bundle of M and s_i the coordinate vector fields $\partial/\partial x^i$.
  >
  > Because we have an explicit description of the manifold structure on the tangent bundle TM, a direct proof is also possible ...

  To use Proposition 12.12 we need to show that the coordinate vector fields $\partial/\partial x^i$ are smooth for all i. Since a direct proof using the manifold structure of TM is given we consider the use of Proposition 12.12 as more of a remark.

Page 151

- > ...a vector field X on a manifold M induces a linear map on the algebra of $C^\infty(M)$ of C^∞ functions on M: for $f \in C^\infty(M)$, define Xf to be the function
  >
  > $$(Xf)(p) = X_p f, \quad p \in M.$$

  We note that the codomain of this linear map is purposely unspecified. In fact, Proposition 14.3 will tell us when the codomain may be restricted to $C^\infty(M)$ (in general it is not). Also, since X_p is defined as a point derivation we are implicitly factoring f through the map $C^\infty(M) \to C_p^\infty(M)$.

- > ...it can be shown that these two descriptions of C^∞ vector fields are equivalent (Problem 19.12) ...

  To prove this equivalence we require some results about differential forms and the exterior derivative of manifolds. For this reason we will not provide the proof here but state that no further results about vector fields are needed so no logical contradictions will occur if one wises to read §18, §19, and prove Problem 19.12 before moving on.

Page 153

- $$c_s(c_t(p)) = c_{s+t}(p)$$

  Here c_t is the counterclockwise rotation matrix through an angle t. Similarly for $c(s)$ and c_{s+t}.

Page 157

- ...Suppose X and Y are smooth vector fields on an open subset U of a manifold M. We view X and Y as derivations on $C^\infty(U)$ ...

  All we mean is that X and Y are standing for the derivations associated to these vector fields since the vector fields X and Y are only defied on an open subset U of M and not on all of M (we don't have the two equivalent definitions for vector fields in this more general setting).

- ...it is easy to check that $[X,Y]_p$ is a derivation of $C_p^\infty(U)$ and is therefore a tangent vector at p ...

  Suppose f and g are in the same germ class at p. Then they agree on some open neighborhood V about p. Therefore Xf and Xg agree on V and so are in the same germ class at p, and similarly for Yf and Yg. Then $X_pYf = X_pYg$ and $Y_pXf = Y_pXg$ so that $[X,Y]_pf = [X,Y]_pg$. This proves $[X,Y]_p : C_p^\infty(U) \to \mathbb{R}$ is a well-defined map. It's $\mathbb{R}$-linear since $[X,Y]$ is a sum and composition of $\mathbb{R}$-linear maps. In fact, it's a point-derivation of $C_p^\infty(U)$:

  $$\begin{aligned}
[X,Y]_p(fg) &= (X_pY - Y_pX)(fg)\\
&= X_pY(fg) - Y_pX(fg)\\
&= X_p((Yf)g + f(Yg)) - Y_p((Xf)g + f(Xg))\\
&= X_p((Yf)g) + X_p(f(Yg)) - Y_p((Xf)g) - Y_p(f(Xg))\\
&= X_p(Yf)g(p) + (Yf)(p)X_p(g) + X_p(f)(Yg)(p) + f(p)X_p(Yg)\\
&\quad - Y_p(Xf)g(p) - (Xf)(p)Y_p(g) - Y_p(f)(Xg)(p) - f(p)Y_p(Xg)\\
&= (X_p(Yf) - Y_p(Xf))g(p) + f(p)(X_p(Yg) - Y_p(Xg))\\
&= ([X,Y]_pf)g(p) + f(p)([X,Y]_pg),
\end{aligned}$$

where the second to last equality holds because the second and seventh terms and third and sixth terms in the third to last line cancel out. All of this proves $[X, Y]_p$ is a tangent vector at p.

Page 158

- > ...the Jacobi identity follows from the same computation as in Exercise 14.11.
  >
  > More generally, if A is any algebra over a field K, then the product
  >
  > $$[x, y] = xy - yx, \quad x, y \in A,$$
  >
  > makes A into a Lie algebra over K ...

Suppose A is an algebra over the field K. Then

$$\begin{aligned}
[ax + by, z] &= (ax + by)z - z(ax + by) \\
&= axz + byz - azx - bzy \\
&= a(xz - zx) + b(yz - zy) \\
&= a[x, z] + b[y, z].
\end{aligned}$$

because A is an algebra. Similarly,

$$\begin{aligned}
[z, ax + by] &= z(ax + by) - (ax + by)z \\
&= azx + bzy - axz - byz \\
&= a(zx - xz) + b(zy - yz) \\
&= a[z, x] + b[z, y].
\end{aligned}$$

This proves the bracket is bilinear. Clearly $[x, y] = -[y, x]$, so the bracket is anticommutative. The Jacobi identity is proved just by expanding the bracket and collecting terms. More precisely, since $\sum_{\text{cyclic}}[x, [y, z]]$ is symmetric in x, y, and z, and the bracket is symmetric in x and y it follows that $\sum_{\text{cyclic}}[x, [y, z]] = 0$ by anticommutivity. Now the space $K^{n \times n}$ of $n \times n$ matrices over K is an algebra over K and the bracket in this space is the same as the product defined above, so $K^{n \times n}$ is a Lie algebra over K.