Asymptotics of Moments of Dirichlet L-series

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I. Motivation. The Riemann zeta-function is a function defined on the half-plane $\Re(s) > 1$ of a complex variable s given by

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}.$$

Riemann showed in 1859 [5] that $\zeta(s)$ can be extended to all of $\mathbb{C}$ by analytic continuation a simple pole at $s = 1$ (with residue 1), satisfies a functional equation, and has trivial zeros at $-2, -4, -6, \ldots$. The celebrated Riemann hypothesis, of critical importance in analytic number theory, is the statement that all nontrivial zeros of $\zeta(s)$ occur on the line $\Re(s) = \frac{1}{2}$. Elegant with far reaching consequences, having resisted all attempts of the finest mathematicians for the last 160 years, and now carrying a one million dollar prize from the Clay Institute, it is arguably the most important open problem in mathematics.

The zeta-function is the prototype of a more general class of functions which we call L-functions. These functions have similar properties (i.e., Euler product expansions and functional equations), and it is conjectured that they should satisfy an analog of the Riemann hypothesis. It is believed that studying the collective behavior of the L-functions is more profitable than studying the zeta-function alone. In the 1980's an idea emerged that it could be useful to tie together a family of related L-functions to create a double, or multiple, Dirichlet series. By studying the analytic properties of the double/multiple Dirichlet series one could then deduce the average behavior of the original family of L-functions.

In 2005, precise conjectures were put forth by Conrey, Farmer, Keating, Rubinstein, and Snaith [2] regarding the asymptotic behavior of certain families of L-functions, namely moments of the corresponding families. These conjectures were motivated, in part, by the so-called Katz-Sarnak philosophy [4]. In 2019, A. Diaconu proposed a refined conjecture for moments of the classical family of quadratic Dirichlet L-functions. His proposal was formulated, for simplicity, in the function field setting, but there is a version of it over number fields. These asymptotic formulas for moments, in the *function field* setting, implies the Riemann hypothesis for each L-function in the family — a famous result of A. Weil [1]. However, no asymptotic formula for high moments of quadratic Dirichlet L-functions is known (even in the function field setting). Due to the strong analogy between phenomena in number fields and in function fields, understanding the refined structure of moments, which goes well-beyond the Riemann hypothesis (at least in the function field setting), should give more insight into the mechanism underlying the Riemann hypothesis in the number field setting (e.g., over the rationals).

II. The problem. The class of L-functions we study are constructed as follows [6]. Let $\mathbb{F}_q$ be a finite field of odd characteristic. Over the rational function field $\mathbb{F}_q(x)$, we have the quadratic character $\chi_d(m) = (d/m)$ for d and m monic polynomials over $\mathbb{F}_q$. The associated

L-function is defined by

$$L(s,\chi_d) := \sum_{m \text{ monic}} \chi_d(m)|m|^{-s} = \prod_{\pi \text{ monic irr.}} (1-\chi_d(\pi)|\pi|^{-s})^{-1} \qquad (\text{for } \Re(s)>1)$$

where $|a| = q^{\deg a}$. For $r, D \in \mathbb{N}$, define the r-th moment by

$$M_r(D) := \sum_{\substack{d \text{ monic sq.-free} \\ \deg d = D}} L(\tfrac{1}{2},\chi_d)^r.$$

We want to study the asymptotics of this average. In 2019, A. Diaconu put forth the following conjecture:

Conjecture. For $r \geq 4$ and small $\epsilon > 0$, we have an asymptotic formula of the form

$$M_r(D) = \frac{q^D}{\zeta(2)} Q_1(D,q) + \sum_{n=2}^{\infty} q^{D(\frac{1}{2}+\frac{1}{2n})} Q_n(D,q) + O_{\epsilon,q,r}(q^{D(\frac{1}{2}+\epsilon)}).$$

My research concerns computing the coefficients $Q_n(D,q)$ explicitly.

III. Computing positive real roots. Assuming only the meromorphic continuation of a certain multiple Dirichlet series, the $Q_n(D,q)$ can be computed by realizing them as contributions parametrized by the *positive real* roots of an underlying Kac-Moody Lie algebra with generalized Cartan matrix

$$\begin{pmatrix} 2 & & & -1 \\ & 2 & & \vdots \\ & & \ddots & -1 \\ -1 & \cdots & -1 & 2 \end{pmatrix}$$

and Weyl group

$$W_r = \langle w_1, \ldots, w_{r+1} \mid w_i^2 = 1, w_{r+1}^2 = 1, (w_i w_j)^2 = 1, (w_i w_{r+1})^3 = 1 \text{ for } 1 \leq i,j \leq r \rangle.$$

This connection stems from the fact that the multiple Dirichlet series we start from satisfies a group of functional equations isomorphic to W_r. All positive roots α in the underlying root system are of the form

$$\alpha = \sum_{i=1}^{r+1} k_i \alpha_i \qquad (k_i \in \mathbb{Z}_{\geq 0} \text{ for } 1 \leq i \leq r+1)$$

where $\{\alpha_i\}$ is the set of simple roots. The difficulty in computing these roots is that the root system associated to this data is *infinite* when $r \geq 4$. The method of computing the positive roots is by an inductive argument, which can be briefly described as follows. First I show that the coefficients k_i of each positive root α with $k_{r+1} \geq 1$ should satisfy $k_i \leq k_{r+1}$ for all i. Thus, using the functional equations of the multiple Dirichlet series, it suffices to compute only the positive real roots with $k_i \leq k_{r+1}/2$ for $1 \leq i \leq r$ — which I call reduced roots. I showed that the reduced roots satisfy the addition bound $k_1 + \cdots + k_r \leq 2k_{r+1} - 1$, from which it follows that the set $\phi_n = \{\alpha \text{ reduced} \mid k_{r+1} = n\}$ can be computed by strong induction over n. In general, the structure of the set ϕ_n is quite complicated, and there seems to be no alternative way of computing it. For example, the following sequence is the size of $|\phi_n|$ for $2 \leq n \leq 20$,

$$1, 1, 2, 3, 3, 6, 5, 9, 9, 16, 12, 26, 19, 33, 25, 52, 39, 74, 47$$

which is not even weakly increasing with n.

IV. Recovering the $Q_n(D, q)$. The next step is to compute the residues at the (simple) poles of a *twisted* multiple Dirichlet series derived from the initial one; the poles of these multiple Dirichlet series are in one-to-one correspondence with the positive real roots discussed above. After a sieving process, we obtain the entire principal part corresponding to each pole of a generating function of moments of L-functions. These principal parts are expressed as a sum parametrized by the sets of roots $\Phi_n = \{\alpha > 0 \mid k_{r+1} = n\}$ $(n \geq 1)$, with the terms in the sum given roughly by

$$\mathfrak{S}_n(\mathrm{z}) := n^{-1} \cdot \sum_{\alpha \in \Phi_n} \sum_{\zeta^n = \pm 1} \Gamma_\infty(\zeta) \left(1 - \zeta q^{(d(\alpha)+1)/2n} \mathrm{z}^{\alpha/n}\right)^{-1} \prod_p S_p(\zeta)$$

where $\mathrm{z} \in \mathbb{C}^{r+1}$, and for $\alpha \in \Phi_n$, we set $d(\alpha) := k_1 + \cdots + k_r + n$, and $\mathrm{z}^{\alpha/n} := z_1^{k_1/n} z_2^{k_2/n} \cdots z_{r+1}$. The function $\Gamma_\infty(\zeta)$ is a certain local contribution at infinity, and $S_p(\zeta)$ is a local contribution at p; both functions depend also on z, but, for simplicity, we suppressed it. There are, however, several difficulties that need to be overcome, the most serious being the fact that the infinite product over p in $\mathfrak{S}_n(\mathrm{z})$ is divergent in the region we are interested in. This difficulty, which was completely resolved, requires a quite subtle regularization combined with an analytic continuation argument. In fact, the divergence of this product, which occurs precisely when $n \geq 2$, explains why the coefficients $Q_n(D, q)$ were completely inaccessible by previous methods. By applying a standard contour integral, I obtained the coefficients $Q_n(D, q)$ from the explicit expression of $\mathfrak{S}_n(\mathrm{z})$.

Finally, I will add that all technical aspects of the argument have been completed, and by April of 2020, a manuscript will be submitted for publication.

References

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