Closed geodesics

Talk by G. Székelyhidi - Notes by H. Twiss
University of Notre Dame

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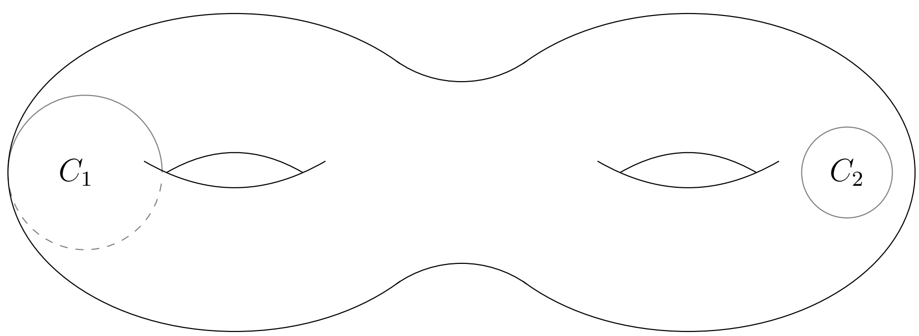
Our setup is as follows: let $M \subset \mathbb{R}^3$ be a closed Riemannian surface, $r : [0,1] \to M$ a geodesic on M with $r(0) = r(1)$. By a geodesic r on M we mean that r is a smooth map of minimal length of equivalently r is a critical point of the length functional. We also say r is closed if $r(0) = r(1)$. Intuitively, a geodesic can be though of as a non-squiggly curve as small scales. It is the generalization of a straight line to arbitrary Riemannian manifolds.

A natural question is the following: does there exist any closed geodesics in M? It is important to note that we are really asking for the existence of a nontrivial closed geodesic in M. For all $p \in M$, the constant geodesic defined by $r(t) = p$ for all $t \in [0,1]$ is a closed geodesic, but not of any real interest. This leads us to our first theorem which happens to be dependent on the fundamental group of M:

Theorem 1. If $\pi_1(M) \neq \{1\}$, then every nontrivial homotopy class contains a closed geodesic

Proof idea. Find the shortest loop in each class, then show this loop is a geodesic. □

Let's expand a little more on how to find the shortest loop in each homotopy class. To find the shortest loop we will utilize the following fact: on M there always exists a length L such that if the length $\ell(r)$ of r satisfies $\ell(r) < L$, then r is homotopic to the trivial loop. To make this fact a little more believable, consider the standard genus two surface with loops C_1 and C_2

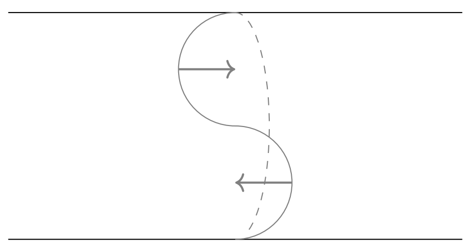


and observe that C^2 is homotopic to the constant loop since C_2 is lying on the "top" of the surface while C_1 is not homotopic to the constant loop since it wraps around the left-most tube. Moreover, any loop that is not homotopic to the constant loop must wrap around either the left or right tube a nonzero net amount of times or wrap around the torus horizontally (like a belt) a nonzero net amount of times. Since this genus two surface is standard, the loops with the smallest length out of the three types described are those that wrap about either the left or right tubes exactly once and they have length 2π (the circumference of a circle). So any loop with length less than 2π doesn't wrap around the torus in any way and must be homotopic to the constant loop. Such a loop is C_2 and not C_1.

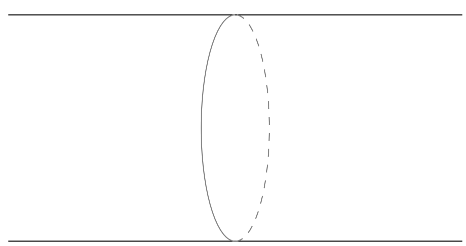
We now want a process which, given a loop r, will shorten the length of r without changing its homotopy class. This process is called curve shortening flow. More precisely, given a loop $r : [0,1] \to M$, we can find a family $r_t : [0,1] \to M$ of loops with $r_0 = r$ such that

$$\frac{dr_t}{dt} = \kappa n \tag{1}$$

where κ is the curvature and n is the unit normal vector. We can then say that r_1 is a closed geodesic homotopic to $r_0 = r$. The idea behind 1 is that we are shortening the length of $r_0 = r$ as efficiently as possible by moving each point of the curve in the direction of the unit normal vector by a factor of the curvature κ. For example, applying curve shortening flow to the loop

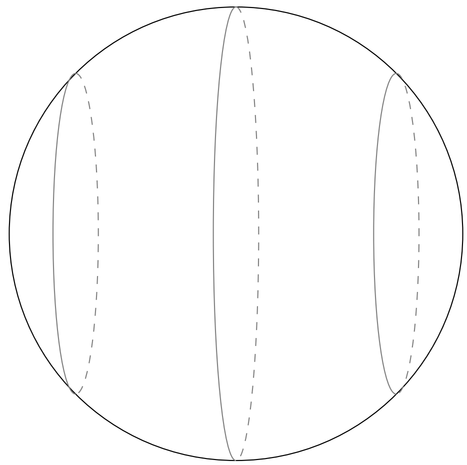


produces the loop



If $\pi_1(M) = \{1\}$, then all loops are trivial and so we can't use Theorem 1 to show that there exist closed geodesics. We can however consider loops of loops or as they are more commonly called: sweepouts. A sweepout is a smooth map $\Phi : [0,1] \to \Omega(M)$[1] such that $\Phi(0) = \Phi(1)$ and the induced map $S^2 \to M$ where we view $S^2 \cong S^1 \times S^1$ is not trivial. For example, a sweep out of the standard sphere looks like

[1] $\Omega(M)$ is the space of all loops on M commonly called the loop space of M.



We may write the induced map more explicitly as

$$\Phi(t,s) : [0,1] \times [0,1] \to M$$

where we understand that the edges of $[0,1] \times [0,1]$ are identified because Φ is closed and $\Phi(t)$ for every $t \in [0,1]$ is a loop. The idea now is to define the width of Φ by $W(\Phi) = \max_{s\in[0,1]} \ell(\Phi(s))$ and try to find the sweepout with minimizing width. Then check that the sweepout with minimizing width contains a geodesic. This method lead to a theorem:

Theorem 2 (Birkhoff 1917)**.** There exists closed geodesics on M.

Proof idea. Show that the width functional cannot be arbitrarily small, and then find a sequence $\{\Phi_k\}_{k\geq 0}$ of sweepouts minimizing the width. Set s_k to such that $W(\Phi_k) = \ell(\Phi_k(s_k))$. Then check that $\lim_{k\to\infty} \Phi_k(s_k)$ is a geodesic. □

Generalizations of this idea to higher dimensions have made there way into current geometry research. For instance, if $M \subset \mathbb{R}^4$ is a closed Riemannian manifold, call a surface $S \subset M$ a minimal surface if it is a critical point of the area functional. Do minimal surfaces exist for M and if so how many are there? We do have two theorems related to this generalization, one very recent and one slightly older:

Theorem 3 (Pitts 1981)**.** If M is a closed Riemannian 3-manifold, then it has at least one minimal surface.

Theorem 4 (Sang 2018, Marques-Neves 2016)**.** If M is a closed Riemannian 3-manifold, it has infinitely many minimal surfaces.

What makes both of these results difficult to prove is that large surfaces can have small areas so checking if a surface is minimal surfaces is a nontrivial task.