Expectany of fixed point permutations using generating functions

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1 Motivation

Given a permutation, the amount of fixed points it has is, loosely-speaking, a measure of how close that permutation is to the identity. Often we want to study this measure in more general settings. So, in this seminar we will

1. Prove that the expectation of fixed points for $\sigma \in S_n$ is 1 for all $n \in \mathbb{N}$.
2. Prove the existence of a function $\varphi_m : S_n \to S_n$ such that the expectation is m for $1 \leq m \leq n$.

2 Preliminaries

Before we move to the main event, we're going to introduce a proposition about a generating function which will serve as the basis of this talk. So without further ado...

Proposition 2.1. In a family $\mathcal{F}$, the number of disconnected structures of weight n that contain exactly a_1 cards of weight 1, a_2 cards of weight 2, etc., where $a_1 + 2a_2 + \cdots = n$, is the coefficient of $(t^n x_1^{a_1} x_2^{a_2} \cdots)/n!$ in the expansion

$$\exp\left\{\sum_{n\in\mathbb{N}} \frac{x_n w_n t^n}{n!}\right\}$$

Proof. It suffices to define the construction of a disconnected structure of this form and show that the number of these disconnected structures is the coefficient of $(t^n x_1^{a_1} x_2^{a_2} \cdots)/n!$ in the egf. We shall build $\mathcal{D}$ in the following way: choose a_1 connected structures from those in weight class $\mathcal{W}_1$, a_2 connected structures from $\mathcal{W}_2$, etc. There are $w_1^{a_1} w_2^{a_2} \cdots / a_1! a_2! \ldots$ possible combinations since the ordering of the connected structures in $\mathcal{D}$ need not be considered. Now choose the labels for the a_1 connected structures in $\mathcal{W}_1$ from $[n]$. We can choose these labels in

$$\binom{n}{1}\binom{n-1}{1}\cdots\binom{n-(n-1)}{1} = \frac{n!}{1!^{a_1}(n-a_1)!}$$

ways. Proceed similarly by choosing labels for the a_2 connected structures in $\mathcal{W}_2$ from $[n] - A_1$, where A_1 is the set of labels chosen for the a_1 connected structures from $\mathcal{W}_1$. For the a_k connected structures in $\mathcal{W}_k$ the number of possible labeling combinations is

$$\frac{(n - a_1 - 2a_2 - 3a_3 \ldots - (k-1)a_{k-1})!}{a_k! k!^{a_k} (n - a_1 - 2a_2 - 3a_3 - \cdots - k a_k)!}$$

Let l_k be this number. Then

$$\prod_k l_k \frac{w_k^{a_k}}{a_k!} = \frac{n! w_1^{a_1} w_2^{a_2} w_3^{a_3} \cdots}{1!^{a_1} 2!^{a_2} 3!^{a_3} \cdots a_1! a_2! a_3! \cdots}$$

which is the desired coefficient of the egf. □

3 Main Event

Now let's get started on the main event. Whenever we mention the cycles of an element in S_n we always mean the cycles in its unique cycle decomposition. We being with a definition:

Definition 3.1. Let $n \in \mathbb{N}$. Then we define

$$C_n(x) = \sum_{\sigma \in S_n} x^{fp(\sigma)}$$

where $fp(\sigma)$ returns the number of fixed points in σ.

Example 3.1. For $n = 3$ we have

$$C_3(x) = \sum_{\sigma \in S_3} x^{fp(\sigma)} = x^3 + 3x + 2$$

That is, there is one element of S_3 with 3 fixed points, 3 with 1 fixed point, and 2 with 0 fixed points.

In particular, $C_n(x)$ is always a monic polynomial of degree n. Now let $C_n(x) : \mathbb{N} \to \mathbb{N}$ so that we may attach a generating function to $C_n(x)$. We will do this by using the generating function in Proposition 2.1.

Theorem 3.1. If $C_n(x)$ is as above, then

$$\sum_{n \in \mathbb{N}} C_n(x) \frac{t^n}{n!} = \exp\left\{ tx + \sum_{n \geq 2} \frac{t^n}{n} \right\}$$

Proof. Consider the egf in Theorem 1 under the permutation family $\mathcal{F}$. Then we are looking at

$$\exp\left\{ \sum_{n \in \mathbb{N}} \frac{x_n t^n}{n} \right\}$$

since $w_n = (n-1)!$. Notice that the summation is the weight class enumerator of $\mathcal{F}$ with attached individual enumerators x_n. This precisely means that the coefficient of $(t^n x_1^{a_1} x_2^{a_2} \cdots)/n!$ is the number of elements in S_n with α_1 fixed points, α_2 2-cycles, etc., where α_i may be 0 for $1 \leq i \leq k$. We only need concern ourselves with fixed points, so let $x_1 = x$ and $x_2 = x_3 = \cdots = 1$. Then the coefficient of $t^n/n!$ becomes $C_n(x)$ and our generating function simplifies to

$$\exp\left\{tx + \sum_{n\geq 2} \frac{t^n}{n}\right\}$$

Apply the exponential formula and the proof is finished. □

We now have a sufficient amount of ammunition to prove the first aim of this talk.

Theorem 3.2. The average number of fixed points in a permutation on n letters is 1.

Proof. Consider the consequence of the exponential formula

$$\sum_{n\in\mathbb{N}} C_n(x)\frac{t^n}{n!} = \exp\left\{tx + \sum_{n\geq 2} \frac{t^n}{n}\right\}$$

Notice that if we take the derivative of $C_n(x)/n!$ and then set $x = 1$, we have the average number of fixed points across all permutations in S_n. So setting $E(n) = C_n'(1)/n!$ gives

$$\begin{aligned} D_x\left(\sum_{n\in\mathbb{N}} C_n(x)\frac{t^n}{n!}\right) &= D_x\left(\exp\left\{tx + \sum_{n\geq 2} \frac{t^n}{n}\right\}\right) \\ \implies \sum_{n\in\mathbb{N}} \frac{C_n'(x)}{n!}t^n &= t\cdot\exp\left\{tx + \sum_{n\geq 2} \frac{t^n}{n}\right\} \\ \implies \sum_{n\in\mathbb{N}} E(n)t^n &= t\cdot\exp\left\{t + \sum_{n\geq 2} \frac{t^n}{n}\right\} \\ &= t\cdot e^{\log\frac{1}{1-t}} \\ &= \frac{t}{1-t} \\ &= \sum_{n\in\mathbb{N}} t^n \end{aligned}$$

Equating coefficients yields $E(n) = 1$. □

Example 3.2.

$$\begin{array}{c} (123) \quad (132) \\ (1)(23) \quad (2)(13) \quad (3)(12) \\ (1)(2)(3) \end{array}$$

After a quick glance, the number of fixed points in S_3 is the same as the size of S_3, so the expectation is 1.

We're now going to need a generalization of $C_n(x)$ in order to prove our second result. In particular, this generalization will let us count the number of m-cycles across elements in S_n.

Definition 3.2. Let $n, m \in \mathbb{N}$. Then

$$C_{n,m}(x) = \sum_{\sigma \in S_n} x^{c_m(\sigma)}$$

where $c_m(\sigma)$ is the number of m-cycles in σ. In particular, $C_{n,1}(x) = C_n(x)$.

We now only need a small proposition before we can prove the second part of this talk.

Proposition 3.1. The number of m-cycles appearing in elements of S_n, namely M_n, is given by $|S_n|/m$ whenever $m \leq n$. In particular, $|S_n| = m \cdot M_n$.

Proof. Applying the exponential formula to $C_{n,m}(x)$ yields

$$\sum_{n \in \mathbb{N}} C_{n,m}(x) \frac{t^n}{n!} = \exp \left\{ \frac{t^m}{m} x + \sum_{\substack{n \in \mathbb{N} \\ n \neq m}} \frac{t^n}{n} \right\}$$

In a similar manner to that of Theorem 3.2, by differentiating with respect to x, setting $x = 1$, and letting $M_n = C'_{n,m}(1)$ we have

$$\begin{aligned}
D_x\left(\sum_{n\in\mathbb{N}} C_{n,m}(x)\frac{t^n}{n!}\right) &= D_x\left(\exp\left\{\frac{t^m}{m}x + \sum_{n\geq 2}\frac{t^n}{n}\right\}\right) \\
\implies \sum_{n\in\mathbb{N}} C'_{n,m}(x)\frac{t^n}{n!} &= \frac{t^m}{m}\cdot\exp\left\{\frac{t^m}{m}x + \sum_{n\geq 2}\frac{t^n}{n}\right\} \\
\implies \sum_{n\in\mathbb{N}} M(n)\frac{t^n}{n!} &= \frac{t^m}{m}\cdot\exp\left\{\frac{t^m}{m} + \sum_{\substack{n\in\mathbb{N}\\ n\neq m}}\frac{t^n}{n}\right\} \\
&= \frac{t^m}{m}e^{\log\frac{1}{1-t}} \\
&= \frac{1}{m}\cdot\frac{t^m}{1-t} \\
&= \sum_{n\geq m}\frac{1}{m}t^n
\end{aligned}$$

Recall that $|S_n| = n!$ and notice $M_n = 0$ if $n < m$. Equating coefficients gives $M_n = |S_n|/m$. □

Looking back at Example 3.2, it can be seen that if we choose any cycle of length m in S_3, the number of those cycles appearing in elements of S_3 is given by $|S_3|/m$. We can now prove the final result of this talk:

Theorem 3.3. Let

$$\varphi_m : S_n \to S_n \qquad \varphi(\sigma) = \sigma^k$$

where $k \in \mathbb{Z}^+$ and m elements in $[n]$ divide k. Then $E(n) = m$ after the map.

Proof. Let $k \in \mathbb{Z}^+$ and let $p_1, \ldots, p_m \in [n]$ such that $p_1, \ldots, p_m$ divide k. Now consider the series of generators

$$C_{p_i,n}(x) = \sum_{\sigma\in S_n} x^{c_{p_i}(\sigma)}$$

for each $i \in [m]$. By Proposition 3.1 these generators tell us there are $|S_n|/p_1$ p_1-cycles, $|S_n|/p_2$ p_2-cycles, etc., appearing in elements of S_n. But these are all the cycles which return the identity after the map since $p_1, \ldots, p_m$ divide k. So the number of fixed points after the map is

$$\sum_{i=1}^{m} \frac{|S_n|}{p_i} p_i = m|S_n|$$

It now follows that $E(n) = m$. $\square$

Example 3.3.
$\varphi(\sigma) = \sigma^2$:

$$(123) \mapsto (132) \quad (132) \mapsto (123)$$
$$(12)(3) \mapsto (1)(2)(3) \quad (13)(2) \mapsto (1)(2)(3) \quad (23)(1) \mapsto (1)(2)(3)$$
$$(1)(2)(3) \mapsto (1)(2)(3)$$

$\varphi(\sigma) = \sigma^6$:

$$(123) \mapsto (1)(2)(3) \quad (132) \mapsto (1)(2)(3)$$
$$(12)(3) \mapsto (1)(2)(3) \quad (13)(2) \mapsto (1)(2)(3) \quad (23)(1) \mapsto (1)(2)(3)$$
$$(1)(2)(3) \mapsto (1)(2)(3)$$