Representation theory and group theory

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Representation theory of finite groups is a deep theory in its own right, but sometimes we can use representations to give some elegant proofs of statements that belong to group theory. Today I'm going to give an example of such a situation. We're going to use some representation theory to prove the following proposition:

Proposition. Let G be a group of order p^2 where p is a prime. Then every irreducible representation of G is one-dimensional. In particular, G is abelian.

Before we can give the proof, we'll need the definition of a representation and give some simple facts.

Definition.

(1) If G is a group, F a field, and V a vector space over F, then a representation is a homomorphism $G \xrightarrow{\varphi} \mathrm{GL}(V)$. We sometimes write φ for brevity.

(2) A subrepresentation of φ is a representation $G \xrightarrow{\varphi|_W} \mathrm{GL}(W)$ with W a vector subspace of V and $\varphi|_W(g) = \varphi(g)|_W$ for every $g \in G$.

(3) The representation φ is irreducible if it has no proper subrepresentation.

(4) The degree of φ is the dimension of V.

Proposition (Facts)**.** Suppose G is a finite group.

(i) If φ is an irreducible representation of G, then the degree of φ divides the order of G.

(ii) The number of irreducible representations of G is equal to the number of conjugacy classes of G.

(iii) Let k be the number of conjugacy classes of G. If n_i is the degree of the i-th irreducible representation, then

$$|G| = \sum_{i=1}^{k} n_i^2.$$

We state these facts without proof. If you take any introductory class in representation theory, then you will very likely prove these facts within the first two weeks of class. They are fairly straightforward, and I would of proved them if I had more time. Now onto the proof.

Proposition. Let G be a group of order p^2 where p is a prime. Then every irreducible representation of G is one-dimensional. In particular, G is abelian.

Proof. Let φ be an irreducible representation of G. By fact (i), the degree of φ is either p^2, p, or 1. We will show it cannot be p^2 or p. If the degree of φ is p^2 then we contradict fact (iii) since $\sum_{i=1}^{k} n_i^2 \geq p^4 > |G|$. If the degree of φ is p, then fact (iii) tells us that φ is the only irreducible representation of G. However the representation $G \xrightarrow{\varphi_{\text{tr}}} \text{GL}(V)$, where V is a one-dimensional vector space, given by $\varphi_{\text{tr}} = \text{id}$ (also known as the trivial representation) is an irreducible representation of degree 1. So, φ cannot be the only irreducible representation and therefore the degree of φ cannot be p. This forces the degree of φ to be 1, and since φ was an arbitrary irreducible representation we conclude every irreducible representation of G is of degree 1. It follows by facts (iii) and (ii) that that the number of conjugacy classes of G is exactly the order of G. This happens if and only if G is abelian, so we are done. $\square$

To conclude, there are also some more classical examples of using representation theory to prove facts about group theory. Two of the most common are proofs of Burnside's theorem and of the Frobenius kernel being a normal subgroup. They mainly use character theory, a subtheory of representations. That concludes my talk, thanks for listening!