Root systems and Hecke algebras

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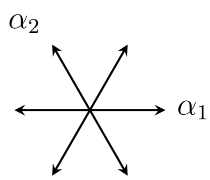
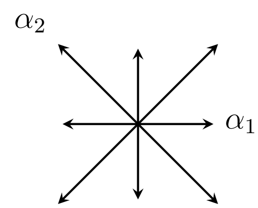
Today we're going to talk about root systems, which are special subset of vectors in a vector space, and their associated Hecke algebras. Our main goal is to showcase the flavor of root systems and Hecke algebras. To get started, what's so special about the following sets of vectors in the plane?

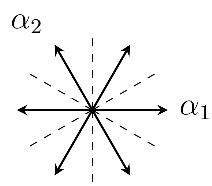


A_2



B_2

The first thing you may notice is both systems span $\mathbb{R}^2$, and this is the first defining property of a root system. These systems satisfy a few more properties to actually be a root system; the most important being that the vectors (which we now call roots) are closed under the reflections perpendicular to them. To understand what this means, consider the diagonal lines:



A_2

If we reflect any root across any of the dashed lines, we get another root. So the dashed lines are what we reflect the roots across, and we say the roots are closed with respect to these reflections (a reflection corresponds to the root perpendicular to the corresponding dashed line). The other system above satisfies this type of closure and so is a root system.

We also labeled certain roots α_1 and α_2. We've done this deliberately because these roots are called the simple roots of the system, and their corresponding reflections s_1 and s_2 are called simple reflections. Simple roots are special in that they form a basis for the root system. For the root system on the right, α_2 is longer than α_1, and so α_2 is said to be a long root and α_1 a short root.

There is an associated group to each set of simple roots of a root system, called the Weyl (pronounced Vile) group W. It is the group of reflections of the root system. It has a Coxeter presentation in terms of the simple roots:

$$W = \langle s_1, \ldots, s_n \mid s_i s_j s_i \cdots = s_j s_i s_j \cdots , s_i^2 = 1 \rangle.$$

Interestingly, we can classify root systems using diagrams. These diagram were introduced by Eugene Dynkin and are rightly called Dynkin diagrams. The Dynkin diagrams of our two roots systems are as follows:

In general, the vertices are represented by the simple reflections, the number of edges between the vertices represent an angle measurement between the corresponding roots, or equivalently the order of the product of the two vertices in W. The arrow always points in the direction of the reflection corresponding to the shorter root.

Our next goal is to show how we may obtain some representations of $\mathrm{GL}_n(\mathbb{F}_q)$. This group is associated to the root system A_{n-1}, so we have the following data:

- A Weyl group $W \cong S_n$ viewed as matrices in $\mathrm{GL}_n(\mathbb{F}_q)$.
- The Borel subgroup $B \subseteq \mathrm{GL}_n(\mathbb{F}_q)$ (consists of invertible upper triangular matrices).
- The Bruhat decomposition: $\mathrm{GL}_n(\mathbb{F}_q) = BWB$ as sets.

With this data we construct a $\mathbb{C}$-algebra $\mathcal{H}_B$ called a Hecke algebra. It's defined as the algebra over $\mathbb{C}$ consisting of B-bi-invariant functions $\phi : \mathrm{GL}_n(\mathbb{F}_q) \to \mathbb{C}$ with convolution as multiplication. It takes some work to show that this algebra has a presentation:

$$\mathcal{H}_B = \langle T_1, \ldots, T_{n-1} \mid T_i T_j T_i \cdots = T_j T_i T_j \cdots , T_i^2 = (q-1)T_i + q \rangle,$$

where the T_is are generators and q is some parameter. This algebra in its B-bi-invariant and presentation forms tell us about some of the representations of $\mathrm{GL}_n(\mathbb{F}_q)$. If we set $q = 1$, then the quadratic relation $T_i^2 = (q-1)T_i + q$ becomes $T_i^2 = 1$, so we get W and therefore as a $\mathbb{C}$-algebra the Hecke algebra is $\mathbb{C}[W]$. So, the Weyl group and hence the root systems are closely linked to the Hecke algebra and the representation theory of $\mathrm{GL}_n(\mathbb{F}_q)$. This the flavor of root systems and Hecke algebras. Thanks for listening!