Symmetric spaces and Cartan's classification

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April 2020

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1 History

It is well-known that Riemann showed that locally there is only one constant curvature geometry. Besides spaces with constant curvature, one of the more natural geometries to study is that of symmetric spaces. We have Elie Cartan alone to thank for the complete classification of symmetric spaces. By the mid 1920s, Cartan had managed to give a complete classification of all symmetric spaces which by today's standards is still a remarkable achievement. Not much happened until the 1950's when people realized a connection between symmetric spaces and holonomy by de Rham's decomposition theorem and Berger's classification of holonomy groups. The latter telling us that almost all holonomy groups occur for symmetric spaces except for a few exceptional cases.

In the following we will introduce symmetric spaces, give a few prototypical examples, and discuss Cartan's classification. We will assume throughout that every Lie algebra is a real Lie algebra unless specified otherwise.

2 Symmetric Spaces

We're going to define symmetric spaces in the usual sense by showing that they are certain types of homogeneous spaces.

We say that a Riemannian manifold M is a symmetric space if for each $p \in M$, there exists an isometry $s_p \in \mathrm{Iso}(M)_p$ such that

$$s_{*,p} : T_pM \to T_pM$$

is the negative of the identity map. The map s_p is called a symmetry at p. We can think of symmetric spaces intuitively in the following way: since geodesics are preserved by isometries, $s_p \circ \gamma$ is a geodesic for all geodesics γ. Since

$$(s_p \circ \gamma)(t) = \gamma(-t),$$

we can think of symmetric spaces as spaces such that at any point there exists a symmetry reversing geodesics through that point. In fact, this observation tells us more. Firstly, if γ is a geodesic defined on $[0, s)$ for some $s > 0$, then we can reflect it by the isometry $s_{\gamma(t)}$ for any $t \in (s/2, s)$ to extend the domain of γ. This proves M is geodesically complete. Secondly, if $p, q \in M$ then we can connect p and q by a geodesic segment γ (since M is geodesically complete). Letting m be the midpoint of γ, we have $s_m(p) = q$ because s_m reverses geodesics through m (in particular it reverse γ). This shows $\mathrm{Iso}(M)^\circ$ acts transitively on M or in other words, M is a homogeneous space.

Conversely, suppose M is a homogeneous space with a symmetry s_p at some $p \in M$. If $q \in M$, let $g \in \mathrm{Iso}(M)^\circ$ be an isometry taking p to q. Then by the chain rule,

$$s_q := g \circ s_p \circ g^{-1}$$

defines a symmetry at q. This proves M is a symmetric space.

We can condense this discussion into the following theorem:

Theorem 2.1. A symmetric space M is precisely a homogeneous space with a symmetry s_p at some $p \in M$.

We can think of symmetric space in another way. Fixing some basepoint $p \in M$ of a symmetric space, the coset space $\mathrm{Iso}(M)^\circ/\mathrm{Iso}(M)_p$ can be identified with M. We state this concretely as a theorem:

Theorem 2.2. Fixing a basepoint $p \in M$,

$$M \cong \mathrm{Iso}(M)^\circ/\mathrm{Iso}(M)_p.$$

Observe that $\mathrm{Iso}(M)^\circ$ is always a connected Lie group, but it is not always the case that $\mathrm{Iso}(M)_p$ is a normal subgroup. Thus $\mathrm{Iso}(M)^\circ/\mathrm{Iso}(M)_p$ need not be a Lie group.

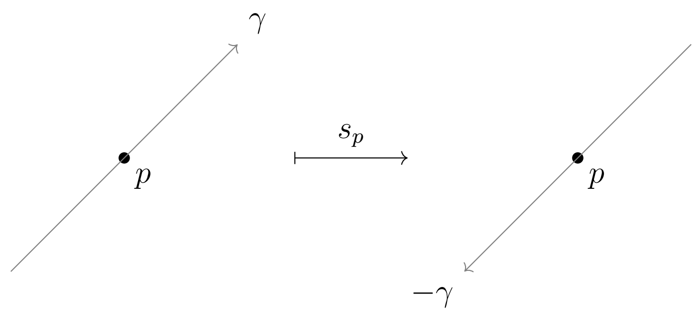
3 Examples of Symmetric Spaces

We are going to give some prototypical examples of symmetric spaces. Our first three examples will be induced from Euclidean space.

Example 3.1 (Euclidean space)**.** Our first example is $\mathbb{R}^n$ with the Euclidean metric. If $p \in \mathbb{R}^n$, then the symmetry s_p at p is defined by

$$s_p(p+v) := p - v.$$

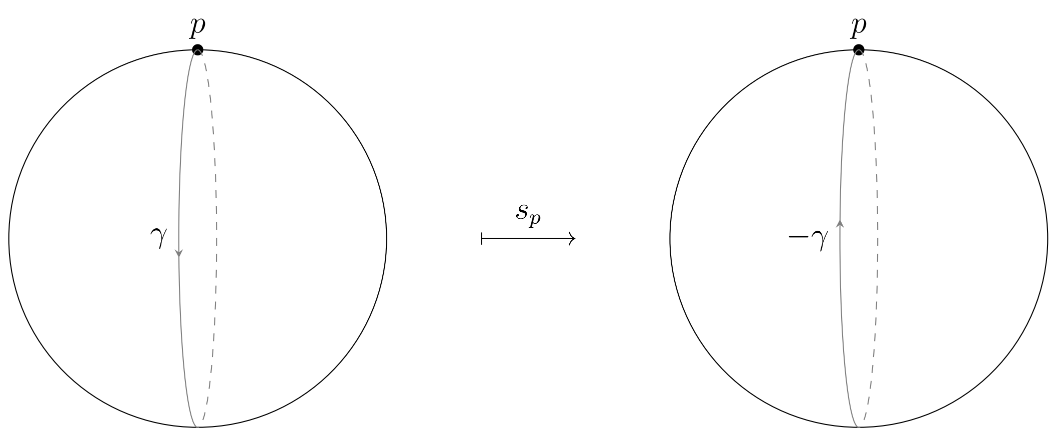
Since any line (i.e., geodesic) through p is of the form $p + tv$ for some $v \in \mathbb{R}^n$, the definition of s_p is easily seen to reverse geodesics through p since $p - tv$ has the opposite direction of $p + tv$. Geometrically:



Example 3.2 (The sphere)**.** Our second example is S^n: the unit sphere in $\mathbb{R}^{n+1}$ with the standard inner product. The symmetry for any $p \in S^n$ is the reflection about the line tp for $t \in \mathbb{R}$ in $\mathbb{R}^{n+1}$. Precisely,

$$s_p(q) := 2\langle q, p\rangle p - q.$$

In terms of great circles on S^n, the symmetry s_p reverse the direction of great circles through p. Geometrically (for S^2):



Example 3.3 (Real hyperbolic space)**.** Our third example is real hyperbolic space $\mathbb{H}^n$. For the definition of $\mathbb{H}^n$, we given $\mathbb{R}^{n+1}$ the Lorentzian scalar product defined by

$$\langle p, q\rangle := \left(\sum_{i=1}^{n} p^i q^i\right) - p^{n+1}q^{n+1},$$

and define $\mathbb{H}^n$ to be

$$\mathbb{H}^n := \{p \in \mathbb{R}^{n+1} \mid \langle p, p\rangle = -1, p^{n+1} > 0\}$$

which is one sheet of the two-sheeted hyperboloid

$$\{p \in \mathbb{R}^{n+1} \mid \langle p, p\rangle = -1\}.$$

The induced scalar product on $T_p\mathbb{H}^n$ is a positive definite inner product making $\mathbb{H}^n$ into a Riemannian manifold. Similar to the example of the sphere, for any $p \in \mathbb{H}^n$, the restriction of

$$s_p(q) := 2\langle q, p\rangle p - q$$

to $\mathbb{H}^n$ is the symmetry through p.

For our last two examples, we'd like to use Theorem 2.1 to show that the orthogonal group $\mathrm{O}(n)$ and, more generally, any compact Lie group are symmetric spaces.

Example 3.4 (Orthogonal group)**.** The orthogonal group $\mathrm{O}(n)$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{R})$ by the regular level set theorem since I is a regular value of the smooth map $f : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{S}(n, \mathbb{R}) \qquad A \mapsto A^T A$ where $\mathrm{S}(n, \mathbb{R})$ is the vector space of symmetric matrices. The Riemannian structure on $\mathrm{O}(n)$ is induced from the scalar product on $\mathbb{R}^n$ by viewing points of $\mathrm{O}(n)$ as elements of $\mathbb{R}^{n\times n}$. In particular,

$$\langle A, B\rangle := \mathrm{trace}(A^T B).$$

We now show $\mathrm{O}(n)$ is a homogeneous space. Clearly $\mathrm{O}(n)$ acts transitively on itself, so it suffices to show left and right multiplication by points of $\mathrm{O}(n)$ are isometries of $\mathrm{O}(n)$. Indeed, if $G \in \mathrm{O}(n)$, then

$$\langle GA, GB\rangle = (GA)^T GB = A^T G^T GB = A^T B = \langle A, B\rangle.$$

Similarly, $\langle AG, BG\rangle = \langle A, B\rangle$. Therefore $\mathrm{O}(n)$ is a homogeneous space. By Theorem 2.1 we need to exhibit a symmetry at the origin I to prove $\mathrm{O}(n)$ is a symmetric space. The most natural map to consider is

$$s_I : \mathrm{O}(n) \to \mathrm{O}(n) \qquad A \mapsto A^T.$$

Clearly s_I is a diffeomorphism preserving the identity. It is an isometry since it preserves the inner product. It's well-known

$$T_I\mathrm{O}(n) = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid A^T = -A\}$$

(as can be checked using curves), so we need to compute

$$s_{I_{*,I}} : T_I\mathrm{O}(n) \to T_I\mathrm{O}(n).$$

Using curves (again) it's a quick check to verify $s_{I_{*,I}}$ is the negative of the identity map. Therefore we have exhibited a symmetry s_I at the origin and Theorem 2.1 implies $\mathrm{O}(n)$ is a symmetric space.

Example 3.5 (Compact Lie groups)**.** We will show any compact Lie group is a symmetric space. Recall that if G is a Lie group, then a biinvariant Riemannian metric on G is a Riemannian metric on G such that left and right translations by elements $g \in G$ are isometries. Since G is compact it exhibits a biinvariant metric (by extending any adjoint invariant inner product in the obvious left invariant manner). Any group acts transitively on itself, so the fact that the metric is biinvariant immediately implies G is homogeneous. Thus by Theorem 2.1 we are left to exhibit a symmetry at the origin. Consider the map

$$s_e : G \to G \qquad g \mapsto g^{-1}.$$

Since G is a Lie group, s_I is a diffeomorphism preserving the identity. Clearly $s_{e_{*,e}}$ preserves the metric. If $g \in G$ is arbitrary, then observe $s_e \circ \ell_g = r_{g^{-1}} \circ s_e$ so by the chain rule

$$s_{e_{*,g}} \circ \ell_{g_{*,e}} = r_{g^{-1}_{*,e}} \circ s_{e_{*,e}}.$$

This shows $s_{e_{*,g}}$ preserves the metric because the other three maps do. Hence s_e is a symmetry at e, proving G is a symmetric space.

Observe that this generalizes our discussion about $\mathrm{O}(n)$ because $\mathrm{O}(n)$ is a compact Lie group and the metric on $\mathrm{O}(n)$ is biinvariant.

4 Curvature and Locally Symmetric Spaces

Another interesting property of symmetric space is that they have a parallel curvature tensor. This almost characterizes symmetric spaces. Generalizing these ideas leads to locally symmetric spaces.

As noted, we have the following theorem for symmetric spaces:

Theorem 4.1. If M is a symmetric space with curvature tensor R, then the curvature tensor is parallel, i.e., $\nabla R = 0$.

Proof sketch. It suffices to prove ∇R is locally parallel. The crux of argument is that if T is a covariant k-tensor in a vector space which is invariant under $-\mathrm{id}$, then $T = (-1)^k T$. If k is odd then necessarily $T = 0$. If M is symmetric, then each point $p \in M$ admits a symmetry s_p which acts by $-\mathrm{id}$ on T_pM. We then check $(\nabla R)_p$ is invariant under $s_{p_*,p}$ and has odd rank. □

We call a Riemannian manifold a locally symmetric space if it has parallel curvature tensor. The following theorem by Cartan justifies the definition:

Theorem 4.2 (Cartan)**.** If M is a locally symmetric space then for each $p \in M$, there is a symmetry at p defined in a neighborhood of p. Moreover, if M is simply connected and complete, M is a symmetric space.

Proof sketch. To prove the first statement, for any $p \in M$, we need to find an isometry defined in a neighborhood of p such that its differential on T_pM is $-\mathrm{id}$. Recall that for small $\epsilon > 0$, the exponential

$$\exp_p : B(0, \epsilon) \to B(p, \epsilon)$$

is a diffeomorphism. So under $\exp_p^{-1}$ we can define $s_p(x) := -x$. In other words we get a diffeomorphism

$$s_p : B(p, \epsilon) \to B(p, \epsilon),$$

and $s_{p_*,p} = -\mathrm{id}$ on T_pM automatically. Now use the fact that the curvature is parallel to prove the metric (under $\exp_p$) at x and $-x$ is the same in $B(0, \epsilon)$. The second statement is proved using an analytic continuation by moving along geodesics. □

The second statement of Theorem 4.2 can be realized as the parallel curvature tensor almost classifies symmetric spaces since being simply connected and completeness are somewhat mild conditions.

5 Effective Orthogonal Symmetric Lie Algebras and the Killing Form

Before discussing Cartan's classification we need to introduce effective orthogonal symmetric Lie algebras and the Killing form. These tools are crucial in the classification of symmetric spaces.

We begin by discussing orthogonal symmetric Lie algebras. An orthogonal symmetric Lie algebra $(\mathfrak{g}, s)$ is a Lie algebra $\mathfrak{g}$ and an involution automorphism s of $\mathfrak{g}$ such that the eigenspace $\mathfrak{u}$ of s corresponding to 1 (i.e., the set of fixed points of s) is a compact Lie subalgebra. Moreover, we say that an orthogonal symmetric Lie algebra is effective if $\mathfrak{u}$ and $Z(\mathfrak{g})$ intersect trivially.

Example 5.1 (Effective orthogonal symmetric Lie algebra)**.** Let $\mathfrak{g} = \mathbb{R}$ and $s = -\mathrm{id}$ so that $\mathfrak{u} = \{0\}$. Then $(\mathfrak{g}, s)$ is an effective orthogonal symmetric Lie algebra.

If $\mathfrak{g}$ is a Lie algebra then we define the Killing form B of $\mathfrak{g}$ over a field $\mathbb{F}$ to be the bilinear form

$$B : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{F} \qquad x \otimes y \mapsto \mathrm{trace}(\mathrm{Ad}(x) \circ \mathrm{Ad}(y)).^1$$

The Killing form happens to be symmetric and satisfies other properties. The sign of the Killing form plays an important role in Cartan's classification so we make the following conventions for orthogonal symmetric Lie algebras $\mathfrak{g}$:

- We say $\mathfrak{g}$ is of compact type if B is negative definite.
- We say $\mathfrak{g}$ is of noncompact type if B is positive definite.
- We say $\mathfrak{g}$ is of flat type if B is identically zero.

It's not difficult to show that Lie algebras of compact type are compact and Lie algebras of noncompact type are noncompact. We also have the following technical theorem, which we will not prove, that will be used in Cartan's classification:

Theorem 5.1. Let $\mathfrak{g}$ be an effective orthogonal symmetric Lie algebra. Then $\mathfrak{g}$ admits the mutually orthogonal decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-$$

where $\mathfrak{g}_0$ is of flat type, $\mathfrak{g}_+$ is of compact type, and $\mathfrak{g}_-$ is of noncompact type.

[1] Usually we take $\mathbb{F} = \mathbb{R}$.

From this viewpoint, every effective orthogonal symmetric Lie algebra is built out from flat type piece, a compact type piece, and a noncompact type piece. The astonishing fact is that this is how (simply connected) symmetric spaces are also built.

If G is a connected Lie group with $K \leq G$ a closed subgroup, we say that (G, K) is a Riemannian symmetric pair if the following two properties are satisfied:

1. $\mathrm{Ad}_G(K) \leq \mathrm{GL}(\mathfrak{g})$ is compact.

2. There exists an involution $\sigma : G \to G$ such that $(G^\sigma)^\circ \subseteq K \subseteq G^\sigma$.

If M is a symmetric space, then $(\mathrm{Iso}(M)^\circ, \mathrm{Iso}(M)_p)$ is a Riemannian symmetric pair with

$$\sigma : \mathrm{Iso}(M)^\circ \to \mathrm{Iso}(M)^\circ \qquad s \mapsto s_p \circ s \circ s_p^{-1}.$$

Conversely, given a Riemannian symmetric pair (G, K), it can be shown that G/K is a symmetric space with respect to any G-invariant Riemannian metric.

Given a Riemannian symmetric pair (G, K), let $\mathfrak{g}$ be the Lie algebra of G and set $s = \sigma_{*,e}$. It turns out that $(\mathfrak{g}, s)$ is a orthogonal symmetric Lie algebra. So, every Riemannian symmetric pair gives rise to a orthogonal symmetric Lie algebra. We say that (G, K) is effective if $Z(G) \cap K$ is a discrete subgroup of G. It's not hard to see that this is equivalent to $(\mathfrak{g}, s)$ being effective. Hence every effective Riemannian symmetric pair gives rise to an effective orthogonal symmetric Lie algebra. By the previous discussion, every symmetric space gives rise to a Riemannian symmetric pair and it turns out that this pair is always effective. Hence we can the following conventions:

- We say an effective Riemannian symmetric pair (G, K) is of flat, compact, or noncompact type if the corresponding effective orthogonal symmetric Lie algebra $(\mathfrak{g}, s)$ is of flat, compact, or noncompact type.

- We say a symmetric space M is of flat, compact, or noncompact type if the corresponding effective orthogonal symmetric Lie algebra is of flat, compact, or noncompact type.

It is the second of these two conventions that we will make use of.

6 Decomposition of Symmetric Spaces

Symmetric spaces also decompose according to compact, noncompact, and Euclidean type. The first observation to make is that the product of symmetric spaces is symmetric since the symmetries of the product are the product of the symmetries. With this observation, Cartan was able to prove the following theorem:

Theorem 6.1. Any simply connected irreducible symmetric space M admits a decomposition

$$M \cong M_0 \times M_+ \times M_-$$

into symmetric spaces where M_0 is of flat type, M_+ is of compact type, and M_- is of noncompact type.

Proof sketch. By Theorem 2.2 M can be identified with the quotient $\mathrm{Iso}(M)^\circ/\mathrm{Iso}(M)_p$ if we fix a basepoint $p \in M$. We know that $(\mathrm{Iso}(M)^\circ, \mathrm{Iso}(M)_p)$ is a Riemannian symmetric pair and thus has associated to it an effective orthogonal symmetric Lie algebra $(\mathfrak{g}, s)$ admitting a decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-$$

where $\mathfrak{g}_0$ is of flat type, $\mathfrak{g}_+$ is of compact type, and $\mathfrak{g}_-$ is of noncompact type. Let $\tilde{G}_0$, $\tilde{G}_+$, and $\tilde{G}_-$ be the Lie groups which are the covering spaces of the Lie groups associated to the Lie algebras above. Let K_0, K_+, and K_- be the Lie groups corresponding to the Lie subalgebras $\mathfrak{u}_0$, $\mathfrak{u}_+$, and $\mathfrak{u}_-$. Then check

$$M \cong (\tilde{G}_0/K_0) \times (\tilde{G}_+/K_+) \times (\tilde{G}_-/K_-)$$

where $\tilde{G}_0/K_0$ is of flat type, $\tilde{G}_+/K_+$ is of compact type, and $\tilde{G}_-/K_-$ is of noncompact type. This finishes the proof. □

Similar to Berger's classification, the assumption that M is simply connected can be made without loss of generality because the universal cover of a symmetric space is a symmetric space. Also, K_0, K_1, or K_2 may not be normal Lie subgroups of $\tilde{G}_0$, $\tilde{G}_+$, and $\tilde{G}_-$ respectively so the corresponding quotients may not be Lie groups themselves. This is prevents M from being a Lie group in general.
If we assume M is irreducible, i.e., not a product of symmetric spaces, then Theorem 6.1 says M is either of compact, noncompact, or Euclidean type. So, it suffices to classify symmetric spaces of compact, noncompact, and Euclidean type. In order to do so we will need an invariant: the rank of a symmetric space.

7 Rank and Classification

Suppose M is an irreducible symmetric space and we have a totally geodesic immersion of Euclidean space into M. Such immersed submanifolds are called flats. We say that a flat is maximal if it is not contained in any larger flat. Basic Lie theory shows that all maximal flats of M are of the same dimension, and we call this common dimension the rank of M. The rank of a symmetric space plays a very important role in Cartan's classification as we now explain.

The rank of an irreducible symmetric space is always at least one with equality if the sectional curvature is positive or negative. If the sectional curvature is positive the space is of compact type, and if the sectional curvature is negative then the space is of noncompact type. For Euclidean type spaces, the rank classifies them completely. The rank of a Euclidean type space is equal to its dimension and this can be used to show that Euclidean type spaces are isometric to Euclidean space of that dimension. So in the notation of Theorem 6.1 $M_0 \cong \mathbb{R}^n$ for some $n \geq 1$. Therefore we are reduces to classifying symmetric spaces of compact and noncompact type. In both cases, we have two classes of symmetric spaces described in terms of Riemannian symmetric pairs (G, K).

Compact type:

- $G = H \times H$ where H is a simply connected compact Lie group and K is the diagonal subgroup.

- G is the complexification of a simply connected noncompact simple Lie group and K is the maximal compact subgroup.

Noncompact type:

- G is a simply connected complex simple Lie group and K is the maximal compact subgroup.

- G is a simply connected noncompact simple Lie group and K is the maximal compact subgroup.

By the classification of Lie groups all such symmetric spaces are also classified, and this finishes Cartan's classification.

References

[1] J. Eschenburg: Lecture Notes on Symmetric Spaces, University of Augsburg.

[2] X. Gao: Symmetric Spaces, University of Illinois, 2014.

[3] P. Holmelin: Symmetric Spaces, Master's Thesis, Lund Institute of Technology, 2005.

[4] P. Petersen: Riemannian Geometry Third Edition, Springer, AG, Switzerland, 2016.