Symmetric spaces and Cartan's classification

Henry Twiss

University of Minnesota

April 2020

Outline

Outline

History

Riemann showed that locally there is only one constant curvature geometry. The most natural geometries to study next are symmetric spaces. Elie Cartan alone classified symmetric spaces.

In the following we will introduce symmetric spaces, give a few prototypical examples, and discuss Cartan's classification.

We will assume throughout that every Lie algebra is a real Lie algebra unless otherwise specified.

History

Riemann showed that locally there is only one constant curvature geometry. The most natural geometries to study next are symmetric spaces. Elie Cartan alone classified symmetric spaces.

In the following we will introduce symmetric spaces, give a few prototypical examples, and discuss Cartan's classification.

We will assume throughout that every Lie algebra is a real Lie algebra unless otherwise specified.

History

Riemann showed that locally there is only one constant curvature geometry. The most natural geometries to study next are symmetric spaces. Elie Cartan alone classified symmetric spaces.

In the following we will introduce symmetric spaces, give a few prototypical examples, and discuss Cartan's classification. We will assume throughout that every Lie algebra is a real Lie algebra unless otherwise specified.

History

Riemann showed that locally there is only one constant curvature geometry. The most natural geometries to study next are symmetric spaces. Elie Cartan alone classified symmetric spaces.

In the following we will introduce symmetric spaces, give a few prototypical examples, and discuss Cartan's classification.

We will assume throughout that every Lie algebra is a real Lie algebra unless otherwise specified.

History

Riemann showed that locally there is only one constant curvature geometry. The most natural geometries to study next are symmetric spaces. Elie Cartan alone classified symmetric spaces.

In the following we will introduce symmetric spaces, give a few prototypical examples, and discuss Cartan's classification. We will assume throughout that every Lie algebra is a real Lie algebra unless otherwise specified.

Outline

Introducing Symmetric Spaces

Definition 2.1

A Riemannian manifold M is a symmetric space if for each $p \in M$, there exists an isometry $s_p \in \mathrm{Iso}(M)_p$ such that

$$s_{*,p} : T_pM \to T_pM$$

is the negative of the identity map. The map s_p is called a symmetry at p.

Geodesics are preserved by isometries, so $s_p \circ \gamma$ is a geodesic for all geodesics γ. Since

$$(s_p \circ \gamma)(t) = \gamma(-t),$$

symmetric spaces as spaces where at any point there exists a symmetry reversing geodesics through that point. This observation tells us more.

Introducing Symmetric Spaces

Definition 2.1

A Riemannian manifold M is a symmetric space if for each $p \in M$, there exists an isometry $s_p \in \mathrm{Iso}(M)_p$ such that

$$s_{*,p} : T_pM \to T_pM$$

is the negative of the identity map. The map s_p is called a symmetry at p.

Geodesics are preserved by isometries, so $s_p \circ \gamma$ is a geodesic for all geodesics γ. Since

$$(s_p \circ \gamma)(t) = \gamma(-t),$$

symmetric spaces as spaces where at any point there exists a symmetry reversing geodesics through that point. This observation tells us more.

Definition 2.1

A Riemannian manifold M is a symmetric space if for each $p \in M$, there exists an isometry $s_p \in \mathrm{Iso}(M)_p$ such that

$$s_{*,p} : T_pM \to T_pM$$

is the negative of the identity map. The map s_p is called a symmetry at p.

Geodesics are preserved by isometries, so $s_p \circ \gamma$ is a geodesic for all geodesics γ. Since

$$(s_p \circ \gamma)(t) = \gamma(-t),$$

symmetric spaces as spaces where at any point there exists a symmetry reversing geodesics through that point. This observation tells us more.

Introducing Symmetric Spaces

Definition 2.1

A Riemannian manifold M is a symmetric space if for each $p \in M$, there exists an isometry $s_p \in \mathrm{Iso}(M)_p$ such that

$$s_{*,p} : T_pM \to T_pM$$

is the negative of the identity map. The map s_p is called a symmetry at p.

Geodesics are preserved by isometries, so $s_p \circ \gamma$ is a geodesic for all geodesics γ. Since

$$(s_p \circ \gamma)(t) = \gamma(-t),$$

symmetric spaces as spaces where at any point there exists a symmetry reversing geodesics through that point. This observation tells us more.

Introducing Symmetric Spaces

Definition 2.1

A Riemannian manifold M is a symmetric space if for each $p \in M$, there exists an isometry $s_p \in \mathrm{Iso}(M)_p$ such that

$$s_{*,p} : T_pM \to T_pM$$

is the negative of the identity map. The map s_p is called a symmetry at p.

Geodesics are preserved by isometries, so $s_p \circ \gamma$ is a geodesic for all geodesics γ. Since

$$(s_p \circ \gamma)(t) = \gamma(-t),$$

symmetric spaces as spaces where at any point there exists a symmetry reversing geodesics through that point. This observation tells us more.

The Homogeneous Description

1. M is geodesically complete: domains of geodesics $\gamma : [0, s)$ are extended by reflecting using symmetries $s_{\gamma(t)}$ for $t \in (s/2, s)$.
2. $\mathrm{Iso}(M)^\circ$ acts transitively on M: connect p and q by a geodesic. Letting m be the midpoint of this geodesic, $s_m(p) = q$.

Definition 2.2

A Riemannian manifold M is a homogeneous space if $\mathrm{Iso}(M)^\circ$ acts transitively on M.

In fact, the second property can be strengthened.

The Homogeneous Description

1. M is geodesically complete: domains of geodesics $\gamma : [0, s)$ are extended by reflecting using symmetries $s_{\gamma(t)}$ for $t \in (s/2, s)$.
2. $\mathrm{Iso}(M)^\circ$ acts transitively on M: connect p and q by a geodesic. Letting m be the midpoint of this geodesic, $s_m(p) = q$.

Definition 2.2

A Riemannian manifold M is a homogeneous space if $\mathrm{Iso}(M)^\circ$ acts transitively on M.

In fact, the second property can be strengthened.

The Homogeneous Description

1. M is geodesically complete: domains of geodesics $\gamma : [0, s)$ are extended by reflecting using symmetries $s_{\gamma(t)}$ for $t \in (s/2, s)$.
2. $\mathrm{Iso}(M)^\circ$ acts transitively on M: connect p and q by a geodesic. Letting m be the midpoint of this geodesic, $s_m(p) = q$.

Definition 2.2

A Riemannian manifold M is a homogeneous space if $\mathrm{Iso}(M)^\circ$ acts transitively on M.

In fact, the second property can be strengthened.

The Homogeneous Description

1. M is geodesically complete: domains of geodesics $\gamma : [0, s)$ are extended by reflecting using symmetries $s_{\gamma(t)}$ for $t \in (s/2, s)$.
2. $\mathrm{Iso}(M)^\circ$ acts transitively on M: connect p and q by a geodesic. Letting m be the midpoint of this geodesic, $s_m(p) = q$.

Definition 2.2

A Riemannian manifold M is a homogeneous space if $\mathrm{Iso}(M)^\circ$ acts transitively on M.

In fact, the second property can be strengthened.

The Homogeneous Description

1. M is geodesically complete: domains of geodesics $\gamma : [0, s)$ are extended by reflecting using symmetries $s_{\gamma(t)}$ for $t \in (s/2, s)$.
2. $\mathrm{Iso}(M)^\circ$ acts transitively on M: connect p and q by a geodesic. Letting m be the midpoint of this geodesic, $s_m(p) = q$.

Definition 2.2

A Riemannian manifold M is a homogeneous space if $\mathrm{Iso}(M)^\circ$ acts transitively on M.

In fact, the second property can be strengthened.

The Homogeneous Description

1. M is geodesically complete: domains of geodesics $\gamma : [0, s)$ are extended by reflecting using symmetries $s_{\gamma(t)}$ for $t \in (s/2, s)$.
2. $\mathrm{Iso}(M)^\circ$ acts transitively on M: connect p and q by a geodesic. Letting m be the midpoint of this geodesic, $s_m(p) = q$.

Definition 2.2

A Riemannian manifold M is a homogeneous space if $\mathrm{Iso}(M)^\circ$ acts transitively on M.

In fact, the second property can be strengthened.

The Homogeneous Description

1. M is geodesically complete: domains of geodesics $\gamma : [0, s)$ are extended by reflecting using symmetries $s_{\gamma(t)}$ for $t \in (s/2, s)$.
2. $\mathrm{Iso}(M)^{\circ}$ acts transitively on M: connect p and q by a geodesic. Letting m be the midpoint of this geodesic, $s_m(p) = q$.

Definition 2.2

A Riemannian manifold M is a homogeneous space if $\mathrm{Iso}(M)^{\circ}$ acts transitively on M.

In fact, the second property can be strengthened.

The Homogeneous Description

Theorem 2.1

A symmetric space M is precisely a homogeneous space with a symmetry s_p at some $p \in M$.

Proof. We are left to show that a homogeneous space with a symmetry is symmetric. Let $g \in \mathrm{Iso}(M)^{\circ}$ be an isometry taking p to q. By the chain rule,

$$s_q := g \circ s_p \circ g^{-1}$$

defines a symmetry at q. □

The Homogeneous Description

Theorem 2.1

A symmetric space M is precisely a homogeneous space with a symmetry s_p at some $p \in M$.

Proof. We are left to show that a homogeneous space with a symmetry is symmetric. Let $g \in \mathrm{Iso}(M)^\circ$ be an isometry taking p to q. By the chain rule,

$$s_q := g \circ s_p \circ g^{-1}$$

defines a symmetry at q. □

The Homogeneous Description

Theorem 2.1

A symmetric space M is precisely a homogeneous space with a symmetry s_p at some $p \in M$.

Proof. We are left to show that a homogeneous space with a symmetry is symmetric. Let $g \in \mathrm{Iso}(M)^\circ$ be an isometry taking p to q. By the chain rule,

$$s_q := g \circ s_p \circ g^{-1}$$

defines a symmetry at q. □

The Homogeneous Description

Theorem 2.1

A symmetric space M is precisely a homogeneous space with a symmetry s_p at some $p \in M$.

Proof. We are left to show that a homogeneous space with a symmetry is symmetric. Let $g \in \mathrm{Iso}(M)^\circ$ be an isometry taking p to q. By the chain rule,

$$s_q := g \circ s_p \circ g^{-1}$$

defines a symmetry at q. □

The Isotropy Description

Theorem 2.2

Fixing a basepoint $p \in M$,

$$M \cong \mathrm{Iso}(M)^\circ / \mathrm{Iso}(M)_p.$$

$\mathrm{Iso}(M)^\circ / \mathrm{Iso}(M)_p$ is not necessarily a Lie group despite $\mathrm{Iso}(M)^\circ$ being a connected Lie group. Indeed, $\mathrm{Iso}(M)_p$ need not be a normal subgroup.

The Isotropy Description

Theorem 2.2

Fixing a basepoint $p \in M$,

$$M \cong \mathrm{Iso}(M)^\circ/\mathrm{Iso}(M)_p.$$

$\mathrm{Iso}(M)^\circ/\mathrm{Iso}(M)_p$ is not necessarily a Lie group despite $\mathrm{Iso}(M)^\circ$ being a connected Lie group. Indeed, $\mathrm{Iso}(M)_p$ need not be a normal subgroup.

The Isotropy Description

Theorem 2.2

Fixing a basepoint $p \in M$,

$$M \cong \mathrm{Iso}(M)^\circ/\mathrm{Iso}(M)_p.$$

$\mathrm{Iso}(M)^\circ/\mathrm{Iso}(M)_p$ is not necessarily a Lie group despite $\mathrm{Iso}(M)^\circ$ being a connected Lie group. Indeed, $\mathrm{Iso}(M)_p$ need not be a normal subgroup.

Outline

Euclidean Space

Endow $\mathbb{R}^n$ with the Euclidean metric. The symmetry s_p at $p \in \mathbb{R}^n$ is defined by

$$s_p(p+v) := p - v.$$

Any line (i.e., geodesic) through p is of the form $p + tv$ for some $v \in \mathbb{R}^n$, so s_p reverse geodesics through p. Geometrically:

Euclidean Space

Endow $\mathbb{R}^n$ with the Euclidean metric. The symmetry s_p at $p \in \mathbb{R}^n$ is defined by

$$s_p(p + v) := p - v.$$

Any line (i.e., geodesic) through p is of the form $p + tv$ for some $v \in \mathbb{R}^n$, so s_p reverse geodesics through p.

Geometrically:

Euclidean Space

Endow $\mathbb{R}^n$ with the Euclidean metric. The symmetry s_p at $p \in \mathbb{R}^n$ is defined by

$$s_p(p + v) := p - v.$$

Any line (i.e., geodesic) through p is of the form $p + tv$ for some $v \in \mathbb{R}^n$, so s_p reverse geodesics through p. Geometrically:

Euclidean Space

Endow $\mathbb{R}^n$ with the Euclidean metric. The symmetry s_p at $p \in \mathbb{R}^n$ is defined by

$$s_p(p + v) := p - v.$$

Any line (i.e., geodesic) through p is of the form $p + tv$ for some $v \in \mathbb{R}^n$, so s_p reverse geodesics through p.

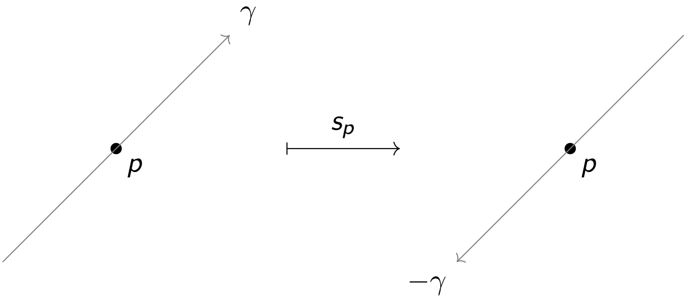
Geometrically:

Euclidean Space

Endow $\mathbb{R}^n$ with the Euclidean metric. The symmetry s_p at $p \in \mathbb{R}^n$ is defined by

$$s_p(p+v) := p - v.$$

Any line (i.e., geodesic) through p is of the form $p + tv$ for some $v \in \mathbb{R}^n$, so s_p reverse geodesics through p. Geometrically:



The Sphere

Endow the unit sphere S^n in $\mathbb{R}^{n+1}$ with the metric induced from the standard inner product. For $p \in S^n$, s_p is the reflection about the line tp for $t \in \mathbb{R}$ in $\mathbb{R}^{n+1}$. Precisely,

$$s_p(q) := 2\langle q, p\rangle p - q.$$

The symmetry s_p reverse the direction of great circles through p. Geometrically (for S^2):

The Sphere

Endow the unit sphere S^n in $\mathbb{R}^{n+1}$ with the metric induced from the standard inner product. For $p \in S^n$, s_p is the reflection about the line tp for $t \in \mathbb{R}$ in $\mathbb{R}^{n+1}$. Precisely,

$$s_p(q) := 2\langle q, p\rangle p - q.$$

The symmetry s_p reverse the direction of great circles through p. Geometrically (for S^2):

The Sphere

Endow the unit sphere S^n in $\mathbb{R}^{n+1}$ with the metric induced from the standard inner product. For $p \in S^n$, s_p is the reflection about the line tp for $t \in \mathbb{R}$ in $\mathbb{R}^{n+1}$. Precisely,

$$s_p(q) := 2\langle q, p\rangle p - q.$$

The symmetry s_p reverse the direction of great circles through p. Geometrically (for S^2):

The Sphere

Endow the unit sphere S^n in $\mathbb{R}^{n+1}$ with the metric induced from the standard inner product. For $p \in S^n$, s_p is the reflection about the line tp for $t \in \mathbb{R}$ in $\mathbb{R}^{n+1}$. Precisely,

$$s_p(q) := 2\langle q, p \rangle p - q.$$

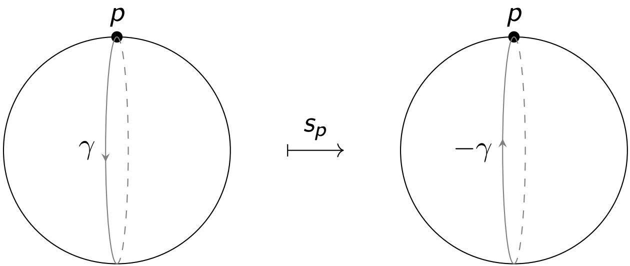
The symmetry s_p reverse the direction of great circles through p. Geometrically (for S^2):

The Sphere

Endow the unit sphere S^n in $\mathbb{R}^{n+1}$ with the metric induced from the standard inner product. For $p \in S^n$, s_p is the reflection about the line tp for $t \in \mathbb{R}$ in $\mathbb{R}^{n+1}$. Precisely,

$$s_p(q) := 2\langle q, p\rangle p - q.$$

The symmetry s_p reverse the direction of great circles through p. Geometrically (for S^2):



Real Hyperbolic Space

To define $\mathbb{H}^n$, give $\mathbb{R}^{n+1}$ the Lorentzian scalar product defined by

$$\langle p, q \rangle := \left(\sum_{i=1}^{n} p^i q^i \right) - p^{n+1} q^{n+1},$$

and define $\mathbb{H}^n$ to be

$$\mathbb{H}^n := \{ p \in \mathbb{R}^{n+1} \mid \langle p, p \rangle = -1, p^{n+1} > 0 \}.$$

The induced scalar product on $T_p\mathbb{H}^n$ for $p \in \mathbb{H}^n$ makes $\mathbb{H}^n$ into a Riemannian manifold. For any $p \in \mathbb{H}^n$, the restriction of

$$s_p(q) := 2\langle q, p \rangle p - q$$

to $\mathbb{H}^n$ is the symmetry through p.

Real Hyperbolic Space

To define $\mathbb{H}^n$, give $\mathbb{R}^{n+1}$ the Lorentzian scalar product defined by

$$\langle p, q \rangle := \left(\sum_{i=1}^{n} p^i q^i \right) - p^{n+1} q^{n+1},$$

and define $\mathbb{H}^n$ to be

$$\mathbb{H}^n := \{ p \in \mathbb{R}^{n+1} \mid \langle p, p \rangle = -1, p^{n+1} > 0 \}.$$

The induced scalar product on $T_p\mathbb{H}^n$ for $p \in \mathbb{H}^n$ makes $\mathbb{H}^n$ into a Riemannian manifold. For any $p \in \mathbb{H}^n$, the restriction of

$$s_p(q) := 2\langle q, p \rangle p - q$$

to $\mathbb{H}^n$ is the symmetry through p.

Real Hyperbolic Space

To define $\mathbb{H}^n$, give $\mathbb{R}^{n+1}$ the Lorentzian scalar product defined by

$$\langle p, q\rangle := \left(\sum_{i=1}^{n} p^i q^i\right) - p^{n+1}q^{n+1},$$

and define $\mathbb{H}^n$ to be

$$\mathbb{H}^n := \{p \in \mathbb{R}^{n+1} \mid \langle p, p\rangle = -1, p^{n+1} > 0\}.$$

The induced scalar product on $T_p\mathbb{H}^n$ for $p \in \mathbb{H}^n$ makes $\mathbb{H}^n$ into a Riemannian manifold. For any $p \in \mathbb{H}^n$, the restriction of

$$s_p(q) := 2\langle q, p\rangle p - q$$

to $\mathbb{H}^n$ is the symmetry through p.

Real Hyperbolic Space

To define $\mathbb{H}^n$, give $\mathbb{R}^{n+1}$ the Lorentzian scalar product defined by

$$\langle p, q \rangle := \left(\sum_{i=1}^{n} p^i q^i \right) - p^{n+1} q^{n+1},$$

and define $\mathbb{H}^n$ to be

$$\mathbb{H}^n := \{ p \in \mathbb{R}^{n+1} \mid \langle p, p \rangle = -1, p^{n+1} > 0 \}.$$

The induced scalar product on $T_p\mathbb{H}^n$ for $p \in \mathbb{H}^n$ makes $\mathbb{H}^n$ into a Riemannian manifold. For any $p \in \mathbb{H}^n$, the restriction of

$$s_p(q) := 2\langle q, p \rangle p - q$$

to $\mathbb{H}^n$ is the symmetry through p.

The Orthogonal Group

We first show $\mathrm{O}(n)$ is a homogeneous space. $\mathrm{O}(n)$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{R})$ by the regular level set theorem. The Riemannian structure on $\mathrm{O}(n)$ is induced from the scalar product on $\mathbb{R}^n$. In particular,

$$\langle A, B \rangle := \mathrm{trace}(A^T B).$$

If $G \in \mathrm{O}(n)$, then

$$\langle GA, GB \rangle = (GA)^T GB = A^T G^T GB = A^T B = \langle A, B \rangle.$$

Similarly, $\langle AG, BG \rangle = \langle A, B \rangle$. Therefore $\mathrm{O}(n)$ is a homogeneous space since $\mathrm{O}(n)$ acts transitively on itself.

The Orthogonal Group

We first show $\mathrm{O}(n)$ is a homogeneous space. $\mathrm{O}(n)$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{R})$ by the regular level set theorem. The Riemannian structure on $\mathrm{O}(n)$ is induced from the scalar product on $\mathbb{R}^n$. In particular,

$$\langle A, B\rangle := \mathrm{trace}(A^T B).$$

If $G \in \mathrm{O}(n)$, then

$$\langle GA, GB\rangle = (GA)^T GB = A^T G^T GB = A^T B = \langle A, B\rangle.$$

Similarly, $\langle AG, BG\rangle = \langle A, B\rangle$. Therefore $\mathrm{O}(n)$ is a homogeneous space since $\mathrm{O}(n)$ acts transitively on itself.

The Orthogonal Group

We first show $\mathrm{O}(n)$ is a homogeneous space. $\mathrm{O}(n)$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{R})$ by the regular level set theorem. The Riemannian structure on $\mathrm{O}(n)$ is induced from the scalar product on $\mathbb{R}^n$. In particular,

$$\langle A, B \rangle := \mathrm{trace}(A^T B).$$

If $G \in \mathrm{O}(n)$, then

$$\langle GA, GB \rangle = (GA)^T GB = A^T G^T GB = A^T B = \langle A, B \rangle.$$

Similarly, $\langle AG, BG \rangle = \langle A, B \rangle$. Therefore $\mathrm{O}(n)$ is a homogeneous space since $\mathrm{O}(n)$ acts transitively on itself.

The Orthogonal Group

We first show $\mathrm{O}(n)$ is a homogeneous space. $\mathrm{O}(n)$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{R})$ by the regular level set theorem. The Riemannian structure on $\mathrm{O}(n)$ is induced from the scalar product on $\mathbb{R}^n$. In particular,

$$\langle A, B\rangle := \mathrm{trace}(A^T B).$$

If $G \in \mathrm{O}(n)$, then

$$\langle GA, GB\rangle = (GA)^T GB = A^T G^T GB = A^T B = \langle A, B\rangle.$$

Similarly, $\langle AG, BG\rangle = \langle A, B\rangle$. Therefore $\mathrm{O}(n)$ is a homogeneous space since $\mathrm{O}(n)$ acts transitively on itself.

The Orthogonal Group

We first show $\mathrm{O}(n)$ is a homogeneous space. $\mathrm{O}(n)$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{R})$ by the regular level set theorem. The Riemannian structure on $\mathrm{O}(n)$ is induced from the scalar product on $\mathbb{R}^n$. In particular,

$$\langle A, B \rangle := \mathrm{trace}(A^T B).$$

If $G \in \mathrm{O}(n)$, then

$$\langle GA, GB \rangle = (GA)^T GB = A^T G^T GB = A^T B = \langle A, B \rangle.$$

Similarly, $\langle AG, BG \rangle = \langle A, B \rangle$. Therefore $\mathrm{O}(n)$ is a homogeneous space since $\mathrm{O}(n)$ acts transitively on itself.

The Orthogonal Group

We first show $\mathrm{O}(n)$ is a homogeneous space. $\mathrm{O}(n)$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{R})$ by the regular level set theorem. The Riemannian structure on $\mathrm{O}(n)$ is induced from the scalar product on $\mathbb{R}^n$. In particular,

$$\langle A, B \rangle := \mathrm{trace}(A^T B).$$

If $G \in \mathrm{O}(n)$, then

$$\langle GA, GB \rangle = (GA)^T GB = A^T G^T GB = A^T B = \langle A, B \rangle.$$

Similarly, $\langle AG, BG \rangle = \langle A, B \rangle$. Therefore $\mathrm{O}(n)$ is a homogeneous space since $\mathrm{O}(n)$ acts transitively on itself.

The Orthogonal Group

We first show $\mathrm{O}(n)$ is a homogeneous space. $\mathrm{O}(n)$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{R})$ by the regular level set theorem. The Riemannian structure on $\mathrm{O}(n)$ is induced from the scalar product on $\mathbb{R}^n$. In particular,

$$\langle A, B \rangle := \mathrm{trace}(A^T B).$$

If $G \in \mathrm{O}(n)$, then

$$\langle GA, GB \rangle = (GA)^T GB = A^T G^T GB = A^T B = \langle A, B \rangle.$$

Similarly, $\langle AG, BG \rangle = \langle A, B \rangle$. Therefore $\mathrm{O}(n)$ is a homogeneous space since $\mathrm{O}(n)$ acts transitively on itself.

The Orthogonal Group

By Theorem 2.1 it suffices to exhibit a symmetry at the origin I. Consider

$$s_I : \mathrm{O}(n) \to \mathrm{O}(n) \qquad A \mapsto A^T.$$

s_I is a isometry preserving the identity. It's well-known (using curves)

$$T_I\mathrm{O}(n) = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid A^T = -A\}.$$

Computing

$$s_{I_{*,I}} : T_I\mathrm{O}(n) \to T_I\mathrm{O}(n)$$

using curves, $s_{I_{*,I}}$ is the negative of the identity map. Thus s_I is a symmetry at I.

The Orthogonal Group

By Theorem 2.1 it suffices to exhibit a symmetry at the origin I. Consider

$$s_I : \mathrm{O}(n) \to \mathrm{O}(n) \qquad A \mapsto A^T.$$

s_I is a isometry preserving the identity. It's well-known (using curves)

$$T_I\mathrm{O}(n) = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid A^T = -A\}.$$

Computing

$$s_{I_{*,I}} : T_I\mathrm{O}(n) \to T_I\mathrm{O}(n)$$

using curves, $s_{I_{*,I}}$ is the negative of the identity map. Thus s_I is a symmetry at I.

The Orthogonal Group

By Theorem 2.1 it suffices to exhibit a symmetry at the origin I. Consider

$$s_I : \mathrm{O}(n) \to \mathrm{O}(n) \qquad A \mapsto A^T.$$

s_I is a isometry preserving the identity. It's well-known (using curves)

$$T_I\mathrm{O}(n) = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid A^T = -A\}.$$

Computing

$$s_{I_{*,I}} : T_I\mathrm{O}(n) \to T_I\mathrm{O}(n)$$

using curves, $s_{I_{*,I}}$ is the negative of the identity map. Thus s_I is a symmetry at I.

The Orthogonal Group

By Theorem 2.1 it suffices to exhibit a symmetry at the origin I. Consider

$$s_I : \mathrm{O}(n) \to \mathrm{O}(n) \qquad A \mapsto A^T.$$

s_I is a isometry preserving the identity. It's well-known (using curves)

$$T_I\mathrm{O}(n) = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid A^T = -A\}.$$

Computing

$$s_{I_{*,I}} : T_I\mathrm{O}(n) \to T_I\mathrm{O}(n)$$

using curves, $s_{I_{*,I}}$ is the negative of the identity map. Thus s_I is a symmetry at I.

The Orthogonal Group

By Theorem 2.1 it suffices to exhibit a symmetry at the origin I. Consider

$$s_I : \mathrm{O}(n) \to \mathrm{O}(n) \qquad A \mapsto A^T.$$

s_I is a isometry preserving the identity. It's well-known (using curves)

$$T_I\mathrm{O}(n) = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid A^T = -A\}.$$

Computing

$$s_{I_{*,I}} : T_I\mathrm{O}(n) \to T_I\mathrm{O}(n)$$

using curves, $s_{I_{*,I}}$ is the negative of the identity map. Thus s_I is a symmetry at I.

The Orthogonal Group

By Theorem 2.1 it suffices to exhibit a symmetry at the origin I. Consider

$$s_I : \mathrm{O}(n) \to \mathrm{O}(n) \qquad A \mapsto A^T.$$

s_I is a isometry preserving the identity. It's well-known (using curves)

$$T_I\mathrm{O}(n) = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid A^T = -A\}.$$

Computing

$$s_{I_{*,I}} : T_I\mathrm{O}(n) \to T_I\mathrm{O}(n)$$

using curves, $s_{I_{*,I}}$ is the negative of the identity map. Thus s_I is a symmetry at I.

Compact Lie Group

Any compact Lie group is a symmetric space. If G is a compact compact Lie group it exhibits a biinvariant metric. G acts transitively on itself, implying G is homogeneous. Consider

$$s_e : G \to G \qquad g \mapsto g^{-1}.$$

s_I is a diffeomorphism preserving the identity, and $s_{e_{*,e}}$ preserves the metric. If $g \in G$ is arbitrary, note $s_e \circ \ell_g = r_{g^{-1}} \circ s_e$. By the chain rule

$$s_{e_{*,g}} \circ \ell_{g_{*,e}} = r_{g^{-1}_{*,e}} \circ s_{e_{*,e}}.$$

So s_e is an isometry, and hence a symmetry at e proving G is a symmetric space.

This generalizes our discussion about $\mathrm{O}(n)$.

Compact Lie Group

Any compact Lie group is a symmetric space. If G is a compact compact Lie group it exhibits a biinvariant metric. G acts transitively on itself, implying G is homogeneous. Consider

$$s_e : G \to G \qquad g \mapsto g^{-1}.$$

s_I is a diffeomorphism preserving the identity, and $s_{e_{*,e}}$ preserves the metric. If $g \in G$ is arbitrary, note $s_e \circ \ell_g = r_{g^{-1}} \circ s_e$. By the chain rule

$$s_{e_{*,g}} \circ \ell_{g_{*,e}} = r_{g^{-1}_{*,e}} \circ s_{e_{*,e}}.$$

So s_e is an isometry, and hence a symmetry at e proving G is a symmetric space.

This generalizes our discussion about $O(n)$.

Compact Lie Group

Any compact Lie group is a symmetric space. If G is a compact compact Lie group it exhibits a biinvariant metric. G acts transitively on itself, implying G is homogeneous.

Consider

$$s_e : G \to G \qquad g \mapsto g^{-1}.$$

s_I is a diffeomorphism preserving the identity, and $s_{e_{*,e}}$ preserves the metric. If $g \in G$ is arbitrary, note $s_e \circ \ell_g = r_{g^{-1}} \circ s_e$. By the chain rule

$$s_{e_{*,g}} \circ \ell_{g_{*,e}} = r_{g^{-1}_{*,e}} \circ s_{e_{*,e}}.$$

So s_e is an isometry, and hence a symmetry at e proving G is a symmetric space.

This generalizes our discussion about $\mathrm{O}(n)$.

Any compact Lie group is a symmetric space. If G is a compact compact Lie group it exhibits a biinvariant metric. G acts transitively on itself, implying G is homogeneous. Consider

$$s_e : G \to G \qquad g \mapsto g^{-1}.$$

s_I is a diffeomorphism preserving the identity, and $s_{e_{*,e}}$ preserves the metric. If $g \in G$ is arbitrary, note $s_e \circ \ell_g = r_{g^{-1}} \circ s_e$. By the chain rule

$$s_{e_{*,g}} \circ \ell_{g_{*,e}} = r_{g^{-1}_{*,e}} \circ s_{e_{*,e}}.$$

So s_e is an isometry, and hence a symmetry at e proving G is a symmetric space.

This generalizes our discussion about $\mathrm{O}(n)$.

Compact Lie Group

Any compact Lie group is a symmetric space. If G is a compact compact Lie group it exhibits a biinvariant metric. G acts transitively on itself, implying G is homogeneous. Consider

$$s_e : G \to G \qquad g \mapsto g^{-1}.$$

s_I is a diffeomorphism preserving the identity, and $s_{e_{*,e}}$ preserves the metric. If $g \in G$ is arbitrary, note $s_e \circ \ell_g = r_{g^{-1}} \circ s_e$. By the chain rule

$$s_{e_{*,g}} \circ \ell_{g_{*,e}} = r_{g^{-1}_{*,e}} \circ s_{e_{*,e}}.$$

So s_e is an isometry, and hence a symmetry at e proving G is a symmetric space.

This generalizes our discussion about $\mathrm{O}(n)$.

Compact Lie Group

Any compact Lie group is a symmetric space. If G is a compact compact Lie group it exhibits a biinvariant metric. G acts transitively on itself, implying G is homogeneous. Consider

$$s_e : G \to G \qquad g \mapsto g^{-1}.$$

s_I is a diffeomorphism preserving the identity, and $s_{e_{*,e}}$ preserves the metric. If $g \in G$ is arbitrary, note $s_e \circ \ell_g = r_{g^{-1}} \circ s_e$. By the chain rule

$$s_{e_{*,g}} \circ \ell_{g_{*,e}} = r_{g^{-1}_{*,e}} \circ s_{e_{*,e}}.$$

So s_e is an isometry, and hence a symmetry at e proving G is a symmetric space.

This generalizes our discussion about $\mathrm{O}(n)$.

Compact Lie Group

Any compact Lie group is a symmetric space. If G is a compact compact Lie group it exhibits a biinvariant metric. G acts transitively on itself, implying G is homogeneous. Consider

$$s_e : G \to G \qquad g \mapsto g^{-1}.$$

s_I is a diffeomorphism preserving the identity, and $s_{e_{*,e}}$ preserves the metric. If $g \in G$ is arbitrary, note $s_e \circ \ell_g = r_{g^{-1}} \circ s_e$. By the chain rule

$$s_{e_{*,g}} \circ \ell_{g_{*,e}} = r_{g^{-1}_{*,e}} \circ s_{e_{*,e}}.$$

So s_e is an isometry, and hence a symmetry at e proving G is a symmetric space.

This generalizes our discussion about $\mathrm{O}(n)$.

Compact Lie Group

Any compact Lie group is a symmetric space. If G is a compact compact Lie group it exhibits a biinvariant metric. G acts transitively on itself, implying G is homogeneous. Consider

$$s_e : G \to G \qquad g \mapsto g^{-1}.$$

s_I is a diffeomorphism preserving the identity, and $s_{e_{*,e}}$ preserves the metric. If $g \in G$ is arbitrary, note $s_e \circ \ell_g = r_{g^{-1}} \circ s_e$. By the chain rule

$$s_{e_{*,g}} \circ \ell_{g_{*,e}} = r_{g_{*,e}^{-1}} \circ s_{e_{*,e}}.$$

So s_e is an isometry, and hence a symmetry at e proving G is a symmetric space.

This generalizes our discussion about $\mathrm{O}(n)$.

Compact Lie Group

Any compact Lie group is a symmetric space. If G is a compact compact Lie group it exhibits a biinvariant metric. G acts transitively on itself, implying G is homogeneous. Consider

$$s_e : G \to G \qquad g \mapsto g^{-1}.$$

s_I is a diffeomorphism preserving the identity, and $s_{e_{*,e}}$ preserves the metric. If $g \in G$ is arbitrary, note $s_e \circ \ell_g = r_{g^{-1}} \circ s_e$. By the chain rule

$$s_{e_{*,g}} \circ \ell_{g_{*,e}} = r_{g^{-1}_{*,e}} \circ s_{e_{*,e}}.$$

So s_e is an isometry, and hence a symmetry at e proving G is a symmetric space.

This generalizes our discussion about $\mathrm{O}(n)$.

Outline

Parallel Curvature Tensor

Symmetric space have parallel curvature tensor.

Theorem 4.1

If M is a symmetric space with curvature tensor R, then the curvature tensor is parallel, i.e., $\nabla R = 0$.

Proof sketch. We prove ∇R is locally parallel. Note that if T is a covariant k-tensor in a vector space which is invariant under $-\mathrm{id}$, then $T = (-1)^k T$. If k is odd then necessarily $T = 0$. If M is symmetric, each point $p \in M$ admits a symmetry s_p. We then check $(\nabla R)_p$ is invariant under $s_{p_{*,p}}$ and has odd rank.

Parallel Curvature Tensor

Symmetric space have parallel curvature tensor.

Theorem 4.1

If M is a symmetric space with curvature tensor R, then the curvature tensor is parallel, i.e., $\nabla R = 0$.

Proof sketch. We prove ∇R is locally parallel. Note that if T is a covariant k-tensor in a vector space which is invariant under $-\mathrm{id}$, then $T = (-1)^k T$. If k is odd then necessarily $T = 0$. If M is symmetric, each point $p \in M$ admits a symmetry s_p. We then check $(\nabla R)_p$ is invariant under $s_{p_{*,p}}$ and has odd rank.

Parallel Curvature Tensor

Symmetric space have parallel curvature tensor.

Theorem 4.1

If M is a symmetric space with curvature tensor R, then the curvature tensor is parallel, i.e., $\nabla R = 0$.

Proof sketch. We prove ∇R is locally parallel. Note that if T is a covariant k-tensor in a vector space which is invariant under $-\mathrm{id}$, then $T = (-1)^k T$. If k is odd then necessarily $T = 0$. If M is symmetric, each point $p \in M$ admits a symmetry s_p. We then check $(\nabla R)_p$ is invariant under $s_{p_{*,p}}$ and has odd rank.

Parallel Curvature Tensor

Symmetric space have parallel curvature tensor.

Theorem 4.1

If M is a symmetric space with curvature tensor R, then the curvature tensor is parallel, i.e., $\nabla R = 0$.

Proof sketch. We prove ∇R is locally parallel. Note that if T is a covariant k-tensor in a vector space which is invariant under $-\mathrm{id}$, then $T = (-1)^k T$. If k is odd then necessarily $T = 0$. If M is symmetric, each point $p \in M$ admits a symmetry s_p. We then check $(\nabla R)_p$ is invariant under $s_{p_{*,p}}$ and has odd rank.

Parallel Curvature Tensor

Symmetric space have parallel curvature tensor.

Theorem 4.1

If M is a symmetric space with curvature tensor R, then the curvature tensor is parallel, i.e., $\nabla R = 0$.

Proof sketch. We prove ∇R is locally parallel. Note that if T is a covariant k-tensor in a vector space which is invariant under $-\mathrm{id}$, then $T = (-1)^k T$. If k is odd then necessarily $T = 0$. If M is symmetric, each point $p \in M$ admits a symmetry s_p. We then check $(\nabla R)_p$ is invariant under $s_{p_{*,p}}$ and has odd rank.

Parallel Curvature Tensor

Symmetric space have parallel curvature tensor.

Theorem 4.1

If M is a symmetric space with curvature tensor R, then the curvature tensor is parallel, i.e., $\nabla R = 0$.

Proof sketch. We prove ∇R is locally parallel. Note that if T is a covariant k-tensor in a vector space which is invariant under $-\mathrm{id}$, then $T = (-1)^k T$. If k is odd then necessarily $T = 0$. If M is symmetric, each point $p \in M$ admits a symmetry s_p. We then check $(\nabla R)_p$ is invariant under $s_{p_{*,p}}$ and has odd rank.

Parallel Curvature Tensor

Symmetric space have parallel curvature tensor.

Theorem 4.1

If M is a symmetric space with curvature tensor R, then the curvature tensor is parallel, i.e., $\nabla R = 0$.

Proof sketch. We prove ∇R is locally parallel. Note that if T is a covariant k-tensor in a vector space which is invariant under $-\mathrm{id}$, then $T = (-1)^k T$. If k is odd then necessarily $T = 0$. If M is symmetric, each point $p \in M$ admits a symmetry s_p. We then check $(\nabla R)_p$ is invariant under $s_{p_{*,p}}$ and has odd rank.

Locally Symmetric Spaces

Definition 4.1

A Riemannian manifold M is a locally symmetric space if it has parallel curvature tensor.

A theorem of Cartan justifies this definition:

Theorem 4.2

If M is a locally symmetric space then for each $p \in M$, there is a symmetry at p defined in a neighborhood of p. Moreover, if M is simply connected and complete, M is a symmetric space.

Locally Symmetric Spaces

Definition 4.1

A Riemannian manifold M is a locally symmetric space if it has parallel curvature tensor.

A theorem of Cartan justifies this definition:

Theorem 4.2

If M is a locally symmetric space then for each $p \in M$, there is a symmetry at p defined in a neighborhood of p. Moreover, if M is simply connected and complete, M is a symmetric space.

Locally Symmetric Spaces

Definition 4.1

A Riemannian manifold M is a locally symmetric space if it has parallel curvature tensor.

A theorem of Cartan justifies this definition:

Theorem 4.2

If M is a locally symmetric space then for each $p \in M$, there is a symmetry at p defined in a neighborhood of p. Moreover, if M is simply connected and complete, M is a symmetric space.

Locally Symmetric Spaces

Definition 4.1

A Riemannian manifold M is a locally symmetric space if it has parallel curvature tensor.

A theorem of Cartan justifies this definition:

Theorem 4.2

If M is a locally symmetric space then for each $p \in M$, there is a symmetry at p defined in a neighborhood of p. Moreover, if M is simply connected and complete, M is a symmetric space.

Locally Symmetric Spaces

Proof sketch. For $\epsilon > 0$, the exponential

$$\exp_p : B(0,\epsilon) \to B(p,\epsilon)$$

is a diffeomorphism. Under $\exp_p^{-1}$ we define $s_p(x) := -x$. This induces a diffeomorphism

$$s_p : B(p,\epsilon) \to B(p,\epsilon),$$

and $s_{p_{*,p}} = -\mathrm{id}$ on T_pM automatically. We use the parallel curvature tensor to prove s_p is an isometry. The second statement is proved using an analytic continuation.

Locally Symmetric Spaces

Proof sketch. For $\epsilon > 0$, the exponential

$$\exp_p : B(0, \epsilon) \to B(p, \epsilon)$$

is a diffeomorphism. Under $\exp_p^{-1}$ we define $s_p(x) := -x$. This induces a diffeomorphism

$$s_p : B(p, \epsilon) \to B(p, \epsilon),$$

and $s_{p_{*,p}} = -\mathrm{id}$ on T_pM automatically. We use the parallel curvature tensor to prove s_p is an isometry. The second statement is proved using an analytic continuation.

Locally Symmetric Spaces

Proof sketch. For $\epsilon > 0$, the exponential

$$\exp_p : B(0, \epsilon) \to B(p, \epsilon)$$

is a diffeomorphism. Under $\exp_p^{-1}$ we define $s_p(x) := -x$. This induces a diffeomorphism

$$s_p : B(p, \epsilon) \to B(p, \epsilon),$$

and $s_{p_{*,p}} = -\mathrm{id}$ on T_pM automatically. We use the parallel curvature tensor to prove s_p is an isometry. The second statement is proved using an analytic continuation.

Locally Symmetric Spaces

Proof sketch. For $\epsilon > 0$, the exponential

$$\exp_p : B(0, \epsilon) \to B(p, \epsilon)$$

is a diffeomorphism. Under $\exp_p^{-1}$ we define $s_p(x) := -x$. This induces a diffeomorphism

$$s_p : B(p, \epsilon) \to B(p, \epsilon),$$

and $s_{p_{*,p}} = -\mathrm{id}$ on T_pM automatically. We use the parallel curvature tensor to prove s_p is an isometry. The second statement is proved using an analytic continuation.

Locally Symmetric Spaces

Proof sketch. For $\epsilon > 0$, the exponential

$$\exp_p : B(0, \epsilon) \to B(p, \epsilon)$$

is a diffeomorphism. Under $\exp_p^{-1}$ we define $s_p(x) := -x$. This induces a diffeomorphism

$$s_p : B(p, \epsilon) \to B(p, \epsilon),$$

and $s_{p_{*,p}} = -\mathrm{id}$ on T_pM automatically. We use the parallel curvature tensor to prove s_p is an isometry. The second statement is proved using an analytic continuation.

Outline

Orthogonal Symmetric Lie Algebras

Definition 5.1

An orthogonal symmetric Lie algebra $(\mathfrak{g}, s)$ is a Lie algebra $\mathfrak{g}$ and an involution automorphism s of $\mathfrak{g}$ such that the eigenspace $\mathfrak{u}$ of s corresponding to 1 (i.e., the set of fixed points of s) is a compact Lie subalgebra. An orthogonal symmetric Lie algebra is effective if $\mathfrak{u}$ and $Z(\mathfrak{g})$ intersect trivially.

As a prototypical example, let $\mathfrak{g} = \mathbb{R}$ and $s = -\text{id}$ so that $\mathfrak{u} = \{0\}$. Then $(\mathfrak{g}, s)$ is an effective orthogonal symmetric Lie algebra.

Orthogonal Symmetric Lie Algebras

Definition 5.1

An orthogonal symmetric Lie algebra $(\mathfrak{g}, s)$ is a Lie algebra $\mathfrak{g}$ and an involution automorphism s of $\mathfrak{g}$ such that the eigenspace $\mathfrak{u}$ of s corresponding to 1 (i.e., the set of fixed points of s) is a compact Lie subalgebra. An orthogonal symmetric Lie algebra is effective if $\mathfrak{u}$ and $Z(\mathfrak{g})$ intersect trivially.

As a prototypical example, let $\mathfrak{g} = \mathbb{R}$ and $s = -\text{id}$ so that $\mathfrak{u} = \{0\}$. Then $(\mathfrak{g}, s)$ is an effective orthogonal symmetric Lie algebra.

Orthogonal Symmetric Lie Algebras

Definition 5.1

An orthogonal symmetric Lie algebra $(\mathfrak{g}, s)$ is a Lie algebra $\mathfrak{g}$ and an involution automorphism s of $\mathfrak{g}$ such that the eigenspace $\mathfrak{u}$ of s corresponding to 1 (i.e., the set of fixed points of s) is a compact Lie subalgebra. An orthogonal symmetric Lie algebra is effective if $\mathfrak{u}$ and $Z(\mathfrak{g})$ intersect trivially.

As a prototypical example, let $\mathfrak{g} = \mathbb{R}$ and $s = -\mathrm{id}$ so that $\mathfrak{u} = \{0\}$. Then $(\mathfrak{g}, s)$ is an effective orthogonal symmetric Lie algebra.

Orthogonal Symmetric Lie Algebras

Definition 5.1

An orthogonal symmetric Lie algebra $(\mathfrak{g}, s)$ is a Lie algebra $\mathfrak{g}$ and an involution automorphism s of $\mathfrak{g}$ such that the eigenspace $\mathfrak{u}$ of s corresponding to 1 (i.e., the set of fixed points of s) is a compact Lie subalgebra. An orthogonal symmetric Lie algebra is effective if $\mathfrak{u}$ and $Z(\mathfrak{g})$ intersect trivially.

As a prototypical example, let $\mathfrak{g} = \mathbb{R}$ and $s = -\mathrm{id}$ so that $\mathfrak{u} = \{0\}$. Then $(\mathfrak{g}, s)$ is an effective orthogonal symmetric Lie algebra.

The Killing Form

Definition 5.2

If $\mathfrak{g}$ is a Lie algebra, we define the Killing form B of $\mathfrak{g}$ over a field $\mathbb{F}$ to be the bilinear form

$$B : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{F} \qquad x \otimes y \mapsto \mathrm{trace}(\mathrm{Ad}(x) \circ \mathrm{Ad}(y)).$$

Usually $\mathbb{F} = \mathbb{R}$. The Killing form is symmetric and satisfies other nice property. The sign of the Killing form plays an important role in Cartan's classification.

The Killing Form

Definition 5.2

If $\mathfrak{g}$ is a Lie algebra, we define the Killing form B of $\mathfrak{g}$ over a field $\mathbb{F}$ to be the bilinear form

$$B : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{F} \qquad x \otimes y \mapsto \mathrm{trace}(\mathrm{Ad}(x) \circ \mathrm{Ad}(y)).$$

Usually $\mathbb{F} = \mathbb{R}$. The Killing form is symmetric and satisfies other nice property. The sign of the Killing form plays an important role in Cartan's classification.

The Killing Form

Definition 5.2

If $\mathfrak{g}$ is a Lie algebra, we define the Killing form B of $\mathfrak{g}$ over a field $\mathbb{F}$ to be the bilinear form

$$B : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{F} \qquad x \otimes y \mapsto \mathrm{trace}(\mathrm{Ad}(x) \circ \mathrm{Ad}(y)).$$

Usually $\mathbb{F} = \mathbb{R}$. The Killing form is symmetric and satisfies other nice property. The sign of the Killing form plays an important role in Cartan's classification.

The Killing Form

Definition 5.2

If $\mathfrak{g}$ is a Lie algebra, we define the Killing form B of $\mathfrak{g}$ over a field $\mathbb{F}$ to be the bilinear form

$$B : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{F} \qquad x \otimes y \mapsto \mathrm{trace}(\mathrm{Ad}(x) \circ \mathrm{Ad}(y)).$$

Usually $\mathbb{F} = \mathbb{R}$. The Killing form is symmetric and satisfies other nice property. The sign of the Killing form plays an important role in Cartan's classification.

Types

Definition 5.3

Let $\mathfrak{g}$ be and orthogonal symmetric Lie algebra.

- We say $\mathfrak{g}$ is of compact type if B is negative definite.
- We say $\mathfrak{g}$ is of noncompact type if B is positive definite.
- We say $\mathfrak{g}$ is of flat type if B is identically zero.

Lie algebras of compact type are compact, and Lie algebras of noncompact type are noncompact.

Types

Definition 5.3

Let $\mathfrak{g}$ be and orthogonal symmetric Lie algebra.

- We say $\mathfrak{g}$ is of compact type if B is negative definite.
- We say $\mathfrak{g}$ is of noncompact type if B is positive definite.
- We say $\mathfrak{g}$ is of flat type if B is identically zero.

Lie algebras of compact type are compact, and Lie algebras of noncompact type are noncompact.

Types

Definition 5.3

Let $\mathfrak{g}$ be and orthogonal symmetric Lie algebra.

- We say $\mathfrak{g}$ is of compact type if B is negative definite.
- We say $\mathfrak{g}$ is of noncompact type if B is positive definite.
- We say $\mathfrak{g}$ is of flat type if B is identically zero.

Lie algebras of compact type are compact, and Lie algebras of noncompact type are noncompact.

Types

Definition 5.3

Let $\mathfrak{g}$ be and orthogonal symmetric Lie algebra.

- We say $\mathfrak{g}$ is of compact type if B is negative definite.
- We say $\mathfrak{g}$ is of noncompact type if B is positive definite.
- We say $\mathfrak{g}$ is of flat type if B is identically zero.

Lie algebras of compact type are compact, and Lie algebras of noncompact type are noncompact.

Types

Definition 5.3

Let $\mathfrak{g}$ be and orthogonal symmetric Lie algebra.

- We say $\mathfrak{g}$ is of compact type if B is negative definite.
- We say $\mathfrak{g}$ is of noncompact type if B is positive definite.
- We say $\mathfrak{g}$ is of flat type if B is identically zero.

Lie algebras of compact type are compact, and Lie algebras of noncompact type are noncompact.

The Natural Decomposition

Definition 5.3 is useful when $\mathfrak{g}$ is effective.

Theorem 5.1

Let $\mathfrak{g}$ be an effective orthogonal symmetric Lie algebra. Then $\mathfrak{g}$ admits the mutually orthogonal decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-$$

where $\mathfrak{g}_0$ is of flat type, $\mathfrak{g}_+$ is of compact type, and $\mathfrak{g}_-$ is of noncompact type.

The astonishing fact is that (simply connected) symmetric spaces decompose this way.

The Natural Decomposition

Definition 5.3 is useful when $\mathfrak{g}$ is effective.

Theorem 5.1

Let $\mathfrak{g}$ be an effective orthogonal symmetric Lie algebra. Then $\mathfrak{g}$ admits the mutually orthogonal decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-$$

where $\mathfrak{g}_0$ is of flat type, $\mathfrak{g}_+$ is of compact type, and $\mathfrak{g}_-$ is of noncompact type.

The astonishing fact is that (simply connected) symmetric spaces decompose this way.

The Natural Decomposition

Definition 5.3 is useful when $\mathfrak{g}$ is effective.

Theorem 5.1

Let $\mathfrak{g}$ be an effective orthogonal symmetric Lie algebra. Then $\mathfrak{g}$ admits the mutually orthogonal decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-$$

where $\mathfrak{g}_0$ is of flat type, $\mathfrak{g}_+$ is of compact type, and $\mathfrak{g}_-$ is of noncompact type.

The astonishing fact is that (simply connected) symmetric spaces decompose this way.

Definition 5.3 is useful when $\mathfrak{g}$ is effective.

Theorem 5.1

Let $\mathfrak{g}$ be an effective orthogonal symmetric Lie algebra. Then $\mathfrak{g}$ admits the mutually orthogonal decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-$$

where $\mathfrak{g}_0$ is of flat type, $\mathfrak{g}_+$ is of compact type, and $\mathfrak{g}_-$ is of noncompact type.

The astonishing fact is that (simply connected) symmetric spaces decompose this way.

Riemannian Symmetric Pairs

Definition 5.4

Let G be a connected Lie group with $K \leq G$ a closed subgroup. We say (G, K) is a Riemannian symmetric pair if the following two properties are satisfied:

1. $\mathrm{Ad}_G(K) \leq \mathrm{GL}(\mathfrak{g})$ is compact.
2. There exists an involution $\sigma : G \to G$ such that $(G^\sigma)^\circ \subseteq K \subseteq G^\sigma$.

If M is a symmetric space, $(\mathrm{Iso}(M)^\circ, \mathrm{Iso}(M)_p)$ is a Riemannian symmetric pair with

$$\sigma : \mathrm{Iso}(M)^\circ \to \mathrm{Iso}(M)^\circ \qquad s \mapsto s_p \circ s \circ s_p^{-1}.$$

Given a Riemannian symmetric pair (G, K), G/K is a symmetric space with respect to any G-invariant Riemannian metric.

Riemannian Symmetric Pairs

Definition 5.4

Let G be a connected Lie group with $K \leq G$ a closed subgroup. We say (G, K) is a Riemannian symmetric pair if the following two properties are satisfied:

1. $\mathrm{Ad}_G(K) \leq \mathrm{GL}(\mathfrak{g})$ is compact.
2. There exists an involution $\sigma : G \to G$ such that $(G^\sigma)^\circ \subseteq K \subseteq G^\sigma$.

If M is a symmetric space, $(\mathrm{Iso}(M)^\circ, \mathrm{Iso}(M)_p)$ is a Riemannian symmetric pair with

$$\sigma : \mathrm{Iso}(M)^\circ \to \mathrm{Iso}(M)^\circ \qquad s \mapsto s_p \circ s \circ s_p^{-1}.$$

Given a Riemannian symmetric pair (G, K), G/K is a symmetric space with respect to any G-invariant Riemannian metric.

Riemannian Symmetric Pairs

Definition 5.4

Let G be a connected Lie group with $K \leq G$ a closed subgroup. We say (G, K) is a Riemannian symmetric pair if the following two properties are satisfied:

1. $\mathrm{Ad}_G(K) \leq \mathrm{GL}(\mathfrak{g})$ is compact.
2. There exists an involution $\sigma : G \to G$ such that $(G^\sigma)^\circ \subseteq K \subseteq G^\sigma$.

If M is a symmetric space, $(\mathrm{Iso}(M)^\circ, \mathrm{Iso}(M)_p)$ is a Riemannian symmetric pair with

$$\sigma : \mathrm{Iso}(M)^\circ \to \mathrm{Iso}(M)^\circ \qquad s \mapsto s_p \circ s \circ s_p^{-1}.$$

Given a Riemannian symmetric pair (G, K), G/K is a symmetric space with respect to any G-invariant Riemannian metric.

Riemannian Symmetric Pairs

Definition 5.4

Let G be a connected Lie group with $K \leq G$ a closed subgroup. We say (G, K) is a Riemannian symmetric pair if the following two properties are satisfied:

1. $\mathrm{Ad}_G(K) \leq \mathrm{GL}(\mathfrak{g})$ is compact.
2. There exists an involution $\sigma : G \to G$ such that $(G^\sigma)^\circ \subseteq K \subseteq G^\sigma$.

If M is a symmetric space, $(\mathrm{Iso}(M)^\circ, \mathrm{Iso}(M)_p)$ is a Riemannian symmetric pair with

$$\sigma : \mathrm{Iso}(M)^\circ \to \mathrm{Iso}(M)^\circ \qquad s \mapsto s_p \circ s \circ s_p^{-1}.$$

Given a Riemannian symmetric pair (G, K), G/K is a symmetric space with respect to any G-invariant Riemannian metric.

Riemannian Symmetric Pairs

Definition 5.4

Let G be a connected Lie group with $K \leq G$ a closed subgroup. We say (G, K) is a Riemannian symmetric pair if the following two properties are satisfied:

1. $\mathrm{Ad}_G(K) \leq \mathrm{GL}(\mathfrak{g})$ is compact.
2. There exists an involution $\sigma : G \to G$ such that $(G^\sigma)^\circ \subseteq K \subseteq G^\sigma$.

If M is a symmetric space, $(\mathrm{Iso}(M)^\circ, \mathrm{Iso}(M)_p)$ is a Riemannian symmetric pair with

$$\sigma : \mathrm{Iso}(M)^\circ \to \mathrm{Iso}(M)^\circ \qquad s \mapsto s_p \circ s \circ s_p^{-1}.$$

Given a Riemannian symmetric pair (G, K), G/K is a symmetric space with respect to any G-invariant Riemannian metric.

Riemannian Symmetric Pairs

Definition 5.4

Let G be a connected Lie group with $K \leq G$ a closed subgroup. We say (G, K) is a Riemannian symmetric pair if the following two properties are satisfied:

1. $\mathrm{Ad}_G(K) \leq \mathrm{GL}(\mathfrak{g})$ is compact.
2. There exists an involution $\sigma : G \to G$ such that $(G^\sigma)^\circ \subseteq K \subseteq G^\sigma$.

If M is a symmetric space, $(\mathrm{Iso}(M)^\circ, \mathrm{Iso}(M)_p)$ is a Riemannian symmetric pair with

$$\sigma : \mathrm{Iso}(M)^\circ \to \mathrm{Iso}(M)^\circ \qquad s \mapsto s_p \circ s \circ s_p^{-1}.$$

Given a Riemannian symmetric pair (G, K), G/K is a symmetric space with respect to any G-invariant Riemannian metric.

Correspondences

Given a Riemannian symmetric pair (G, K), let $\mathfrak{g}$ be the Lie algebra of G and set $s = \sigma_{*,e}$. Then $(\mathfrak{g}, s)$ is a orthogonal symmetric Lie algebra.

Definition 5.5

A Riemannian symmetric pair (G, K) is effective if $Z(G) \cap K$ is a discrete subgroup of G.

This is equivalent to $(\mathfrak{g}, s)$ being effective. We know every symmetric space gives rise to a Riemannian symmetric pair; this pair is always effective.

Correspondences

Given a Riemannian symmetric pair (G, K), let $\mathfrak{g}$ be the Lie algebra of G and set $s = \sigma_{*,e}$. Then $(\mathfrak{g}, s)$ is a orthogonal symmetric Lie algebra.

Definition 5.5

A Riemannian symmetric pair (G, K) is effective if $Z(G) \cap K$ is a discrete subgroup of G.

This is equivalent to $(\mathfrak{g}, s)$ being effective. We know every symmetric space gives rise to a Riemannian symmetric pair; this pair is always effective.

Correspondences

Given a Riemannian symmetric pair (G, K), let $\mathfrak{g}$ be the Lie algebra of G and set $s = \sigma_{*,e}$. Then $(\mathfrak{g}, s)$ is a orthogonal symmetric Lie algebra.

Definition 5.5

A Riemannian symmetric pair (G, K) is effective if $Z(G) \cap K$ is a discrete subgroup of G.

This is equivalent to $(\mathfrak{g}, s)$ being effective. We know every symmetric space gives rise to a Riemannian symmetric pair; this pair is always effective.

Correspondences

Given a Riemannian symmetric pair (G, K), let $\mathfrak{g}$ be the Lie algebra of G and set $s = \sigma_{*,e}$. Then $(\mathfrak{g}, s)$ is a orthogonal symmetric Lie algebra.

Definition 5.5

A Riemannian symmetric pair (G, K) is effective if $Z(G) \cap K$ is a discrete subgroup of G.

This is equivalent to $(\mathfrak{g}, s)$ being effective. We know every symmetric space gives rise to a Riemannian symmetric pair; this pair is always effective.

Correspondences

Given a Riemannian symmetric pair (G, K), let $\mathfrak{g}$ be the Lie algebra of G and set $s = \sigma_{*,e}$. Then $(\mathfrak{g}, s)$ is a orthogonal symmetric Lie algebra.

Definition 5.5

A Riemannian symmetric pair (G, K) is effective if $Z(G) \cap K$ is a discrete subgroup of G.

This is equivalent to $(\mathfrak{g}, s)$ being effective. We know every symmetric space gives rise to a Riemannian symmetric pair; this pair is always effective.

Correspondences

Given a Riemannian symmetric pair (G, K), let $\mathfrak{g}$ be the Lie algebra of G and set $s = \sigma_{*,e}$. Then $(\mathfrak{g}, s)$ is a orthogonal symmetric Lie algebra.

Definition 5.5

A Riemannian symmetric pair (G, K) is effective if $Z(G) \cap K$ is a discrete subgroup of G.

This is equivalent to $(\mathfrak{g}, s)$ being effective. We know every symmetric space gives rise to a Riemannian symmetric pair; this pair is always effective.

Types Revisited

Definition 5.6

1. An effective Riemannian symmetric pair (G, K) is of flat, compact, or noncompact type if the corresponding effective orthogonal symmetric Lie algebra $(\mathfrak{g}, s)$ is of flat, compact, or noncompact type.
2. A symmetric space M is of flat, compact, or noncompact type if the corresponding effective orthogonal symmetric Lie algebra is of flat, compact, or noncompact type.

It is the second of these two conventions that we will make use of.

Types Revisited

Definition 5.6

1. An effective Riemannian symmetric pair (G, K) is of flat, compact, or noncompact type if the corresponding effective orthogonal symmetric Lie algebra $(\mathfrak{g}, s)$ is of flat, compact, or noncompact type.
2. A symmetric space M is of flat, compact, or noncompact type if the corresponding effective orthogonal symmetric Lie algebra is of flat, compact, or noncompact type.

It is the second of these two conventions that we will make use of.

Types Revisited

Definition 5.6

1. An effective Riemannian symmetric pair (G, K) is of flat, compact, or noncompact type if the corresponding effective orthogonal symmetric Lie algebra $(\mathfrak{g}, s)$ is of flat, compact, or noncompact type.
2. A symmetric space M is of flat, compact, or noncompact type if the corresponding effective orthogonal symmetric Lie algebra is of flat, compact, or noncompact type.

It is the second of these two conventions that we will make use of.

Outline

Symmetric Space Decomposition

Observe that products of symmetric spaces are symmetric. With this observation, Cartan was able to prove the following theorem:

Theorem 6.1

A simply connected symmetric space M admits a decomposition

$$M \cong M_0 \times M_+ \times M_-$$

into symmetric spaces where M_0 is of flat type, M_+ is of compact type, and M_- is of noncompact type.

Symmetric Space Decomposition

Observe that products of symmetric spaces are symmetric. With this observation, Cartan was able to prove the following theorem:

Theorem 6.1

A simply connected symmetric space M admits a decomposition

$$M \cong M_0 \times M_+ \times M_-$$

into symmetric spaces where M_0 is of flat type, M_+ is of compact type, and M_- is of noncompact type.

Symmetric Space Decomposition

Observe that products of symmetric spaces are symmetric. With this observation, Cartan was able to prove the following theorem:

Theorem 6.1

A simply connected symmetric space M admits a decomposition

$$M \cong M_0 \times M_+ \times M_-$$

into symmetric spaces where M_0 is of flat type, M_+ is of compact type, and M_- is of noncompact type.

Decomposition of Symmetric Spaces

Proof sketch. By Theorem 2.2 $M \cong \mathrm{Iso}(M)^\circ/\mathrm{Iso}(M)_p$ by fixing a basepoint $p \in M$. Recall $(\mathrm{Iso}(M)^\circ, \mathrm{Iso}(M)_p)$ is a Riemannian symmetric pair. Let $(\mathfrak{g}, s)$ be the corresponding effective orthogonal symmetric Lie algebra. By Theorem 5.1

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-.$$

Let $\tilde{G}_0$, $\tilde{G}_+$, and $\tilde{G}_-$ be the Lie groups which are the covering spaces of the Lie groups associated to the Lie algebras above. Let K_0, K_+, and K_- be the Lie groups corresponding to the Lie subalgebras $\mathfrak{u}_0$, $\mathfrak{u}_+$, and $\mathfrak{u}_-$. Then check

$$M \cong (\tilde{G}_0/K_0) \times (\tilde{G}_+/K_+) \times (\tilde{G}_-/K_-),$$

where $\tilde{G}_0/K_0$ is of flat type, $\tilde{G}_+/K_+$ is of compact type, and $\tilde{G}_-/K_-$ is of noncompact type.

Decomposition of Symmetric Spaces

Proof sketch. By Theorem 2.2 $M \cong \mathrm{Iso}(M)^\circ/\mathrm{Iso}(M)_p$ by fixing a basepoint $p \in M$. Recall $(\mathrm{Iso}(M)^\circ, \mathrm{Iso}(M)_p)$ is a Riemannian symmetric pair. Let $(\mathfrak{g}, s)$ be the corresponding effective orthogonal symmetric Lie algebra. By Theorem 5.1

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-.$$

Let $\tilde{G}_0$, $\tilde{G}_+$, and $\tilde{G}_-$ be the Lie groups which are the covering spaces of the Lie groups associated to the Lie algebras above. Let K_0, K_+, and K_- be the Lie groups corresponding to the Lie subalgebras $\mathfrak{u}_0$, $\mathfrak{u}_+$, and $\mathfrak{u}_-$. Then check

$$M \cong (\tilde{G}_0/K_0) \times (\tilde{G}_+/K_+) \times (\tilde{G}_-/K_-),$$

where $\tilde{G}_0/K_0$ is of flat type, $\tilde{G}_+/K_+$ is of compact type, and $\tilde{G}_-/K_-$ is of noncompact type.

Decomposition of Symmetric Spaces

Proof sketch. By Theorem 2.2 $M \cong \mathrm{Iso}(M)^\circ/\mathrm{Iso}(M)_p$ by fixing a basepoint $p \in M$. Recall $(\mathrm{Iso}(M)^\circ, \mathrm{Iso}(M)_p)$ is a Riemannian symmetric pair. Let $(\mathfrak{g}, s)$ be the corresponding effective orthogonal symmetric Lie algebra. By Theorem 5.1

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-.$$

Let $\tilde{G}_0$, $\tilde{G}_+$, and $\tilde{G}_-$ be the Lie groups which are the covering spaces of the Lie groups associated to the Lie algebras above. Let K_0, K_+, and K_- be the Lie groups corresponding to the Lie subalgebras $\mathfrak{u}_0$, $\mathfrak{u}_+$, and $\mathfrak{u}_-$. Then check

$$M \cong (\tilde{G}_0/K_0) \times (\tilde{G}_+/K_+) \times (\tilde{G}_-/K_-),$$

where $\tilde{G}_0/K_0$ is of flat type, $\tilde{G}_+/K_+$ is of compact type, and $\tilde{G}_-/K_-$ is of noncompact type.

Decomposition of Symmetric Spaces

Proof sketch. By Theorem 2.2 $M \cong \mathrm{Iso}(M)^\circ/\mathrm{Iso}(M)_p$ by fixing a basepoint $p \in M$. Recall $(\mathrm{Iso}(M)^\circ, \mathrm{Iso}(M)_p)$ is a Riemannian symmetric pair. Let $(\mathfrak{g}, s)$ be the corresponding effective orthogonal symmetric Lie algebra. By Theorem 5.1

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-.$$

Let $\tilde{G}_0$, $\tilde{G}_+$, and $\tilde{G}_-$ be the Lie groups which are the covering spaces of the Lie groups associated to the Lie algebras above. Let K_0, K_+, and K_- be the Lie groups corresponding to the Lie subalgebras $\mathfrak{u}_0$, $\mathfrak{u}_+$, and $\mathfrak{u}_-$. Then check

$$M \cong (\tilde{G}_0/K_0) \times (\tilde{G}_+/K_+) \times (\tilde{G}_-/K_-),$$

where $\tilde{G}_0/K_0$ is of flat type, $\tilde{G}_+/K_+$ is of compact type, and $\tilde{G}_-/K_-$ is of noncompact type.

Decomposition of Symmetric Spaces

Proof sketch. By Theorem 2.2 $M \cong \mathrm{Iso}(M)^\circ/\mathrm{Iso}(M)_p$ by fixing a basepoint $p \in M$. Recall $(\mathrm{Iso}(M)^\circ, \mathrm{Iso}(M)_p)$ is a Riemannian symmetric pair. Let $(\mathfrak{g}, s)$ be the corresponding effective orthogonal symmetric Lie algebra. By Theorem 5.1

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-.$$

Let $\tilde{G}_0$, $\tilde{G}_+$, and $\tilde{G}_-$ be the Lie groups which are the covering spaces of the Lie groups associated to the Lie algebras above. Let K_0, K_+, and K_- be the Lie groups corresponding to the Lie subalgebras $\mathfrak{u}_0$, $\mathfrak{u}_+$, and $\mathfrak{u}_-$. Then check

$$M \cong (\tilde{G}_0/K_0) \times (\tilde{G}_+/K_+) \times (\tilde{G}_-/K_-),$$

where $\tilde{G}_0/K_0$ is of flat type, $\tilde{G}_+/K_+$ is of compact type, and $\tilde{G}_-/K_-$ is of noncompact type.

Decomposition of Symmetric Spaces

Proof sketch. By Theorem 2.2 $M \cong \mathrm{Iso}(M)^\circ/\mathrm{Iso}(M)_p$ by fixing a basepoint $p \in M$. Recall $(\mathrm{Iso}(M)^\circ, \mathrm{Iso}(M)_p)$ is a Riemannian symmetric pair. Let $(\mathfrak{g}, s)$ be the corresponding effective orthogonal symmetric Lie algebra. By Theorem 5.1

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-.$$

Let $\tilde{G}_0$, $\tilde{G}_+$, and $\tilde{G}_-$ be the Lie groups which are the covering spaces of the Lie groups associated to the Lie algebras above. Let K_0, K_+, and K_- be the Lie groups corresponding to the Lie subalgebras $\mathfrak{u}_0$, $\mathfrak{u}_+$, and $\mathfrak{u}_-$. Then check

$$M \cong (\tilde{G}_0/K_0) \times (\tilde{G}_+/K_+) \times (\tilde{G}_-/K_-),$$

where $\tilde{G}_0/K_0$ is of flat type, $\tilde{G}_+/K_+$ is of compact type, and $\tilde{G}_-/K_-$ is of noncompact type.

Comments

The assumption M is simply connected can be made without loss of generality. If we assume M is irreducible, i.e., not a product of symmetric spaces, then Theorem 6.1 says M is either of compact, noncompact, or Euclidean type. So, it suffices to classify symmetric spaces of these types. In order to do so we will need an invariant: the rank of a symmetric space.

Comments

The assumption M is simply connected can be made without loss of generality. If we assume M is irreducible, i.e., not a product of symmetric spaces, then Theorem 6.1 says M is either of compact, noncompact, or Euclidean type. So, it suffices to classify symmetric spaces of these types. In order to do so we will need an invariant: the rank of a symmetric space.

Comments

The assumption M is simply connected can be made without loss of generality. If we assume M is irreducible, i.e., not a product of symmetric spaces, then Theorem 6.1 says M is either of compact, noncompact, or Euclidean type. So, it suffices to classify symmetric spaces of these types. In order to do so we will need an invariant: the rank of a symmetric space.

Comments

The assumption M is simply connected can be made without loss of generality. If we assume M is irreducible, i.e., not a product of symmetric spaces, then Theorem 6.1 says M is either of compact, noncompact, or Euclidean type. So, it suffices to classify symmetric spaces of these types. In order to do so we will need an invariant: the rank of a symmetric space.

Comments

The assumption M is simply connected can be made without loss of generality. If we assume M is irreducible, i.e., not a product of symmetric spaces, then Theorem 6.1 says M is either of compact, noncompact, or Euclidean type. So, it suffices to classify symmetric spaces of these types. In order to do so we will need an invariant: the rank of a symmetric space.

Outline

Flats and Rank

Definition 7.1

Suppose M is an irreducible symmetric space. A totally geodesic immersion of $\mathbb{R}^n$ into M is called a flat. A flat is maximal if it is not contained in any larger flat.

Basic Lie theory shows that all maximal flats of M are of the same dimension.

Definition 7.2

The rank of and irreducible symmetric space M is the dimension of any maximal flat.

The rank of a symmetric space plays a very important role in Cartan's classification

Flats and Rank

Definition 7.1

Suppose M is an irreducible symmetric space. A totally geodesic immersion of $\mathbb{R}^n$ into M is called a flat. A flat is maximal if it is not contained in any larger flat.

Basic Lie theory shows that all maximal flats of M are of the same dimension.

Definition 7.2

The rank of and irreducible symmetric space M is the dimension of any maximal flat.

The rank of a symmetric space plays a very important role in Cartan's classification

Flats and Rank

Definition 7.1

Suppose M is an irreducible symmetric space. A totally geodesic immersion of $\mathbb{R}^n$ into M is called a flat. A flat is maximal if it is not contained in any larger flat.

Basic Lie theory shows that all maximal flats of M are of the same dimension.

Definition 7.2

The rank of and irreducible symmetric space M is the dimension of any maximal flat.

The rank of a symmetric space plays a very important role in Cartan's classification

Flats and Rank

Definition 7.1

Suppose M is an irreducible symmetric space. A totally geodesic immersion of $\mathbb{R}^n$ into M is called a flat. A flat is maximal if it is not contained in any larger flat.

Basic Lie theory shows that all maximal flats of M are of the same dimension.

Definition 7.2

The rank of and irreducible symmetric space M is the dimension of any maximal flat.

The rank of a symmetric space plays a very important role in Cartan's classification

Flats and Rank

Definition 7.1

Suppose M is an irreducible symmetric space. A totally geodesic immersion of $\mathbb{R}^n$ into M is called a flat. A flat is maximal if it is not contained in any larger flat.

Basic Lie theory shows that all maximal flats of M are of the same dimension.

Definition 7.2

The rank of and irreducible symmetric space M is the dimension of any maximal flat.

The rank of a symmetric space plays a very important role in Cartan's classification

Flats and Rank

Definition 7.1

Suppose M is an irreducible symmetric space. A totally geodesic immersion of $\mathbb{R}^n$ into M is called a flat. A flat is maximal if it is not contained in any larger flat.

Basic Lie theory shows that all maximal flats of M are of the same dimension.

Definition 7.2

The rank of and irreducible symmetric space M is the dimension of any maximal flat.

The rank of a symmetric space plays a very important role in Cartan's classification

Rank and Sectional Curvature

The rank of is at least one with equality if the sectional curvature is positive or negative. If the sectional curvature is positive the space is of compact type, and if the sectional curvature is negative the space is of noncompact type.

The rank of a Euclidean type space is equal to its dimension. This implies Euclidean type spaces are isometric to Euclidean space of that dimension.

Therefore we are reduces to classifying symmetric spaces of compact and noncompact type.

In both cases, we have two classes of symmetric spaces described in terms of Riemannian symmetric pairs (G, K).

Rank and Sectional Curvature

The rank of is at least one with equality if the sectional curvature is positive or negative. If the sectional curvature is positive the space is of compact type, and if the sectional curvature is negative the space is of noncompact type.

The rank of a Euclidean type space is equal to its dimension. This implies Euclidean type spaces are isometric to Euclidean space of that dimension.

Therefore we are reduces to classifying symmetric spaces of compact and noncompact type.

In both cases, we have two classes of symmetric spaces described in terms of Riemannian symmetric pairs (G, K).

Rank and Sectional Curvature

The rank of is at least one with equality if the sectional curvature is positive or negative. If the sectional curvature is positive the space is of compact type, and if the sectional curvature is negative the space is of noncompact type.

The rank of a Euclidean type space is equal to its dimension.

This implies Euclidean type spaces are isometric to Euclidean space of that dimension.

Therefore we are reduces to classifying symmetric spaces of compact and noncompact type.

In both cases, we have two classes of symmetric spaces described in terms of Riemannian symmetric pairs (G, K).

Rank and Sectional Curvature

The rank of is at least one with equality if the sectional curvature is positive or negative. If the sectional curvature is positive the space is of compact type, and if the sectional curvature is negative the space is of noncompact type.

The rank of a Euclidean type space is equal to its dimension. This implies Euclidean type spaces are isometric to Euclidean space of that dimension.

Therefore we are reduces to classifying symmetric spaces of compact and noncompact type.

In both cases, we have two classes of symmetric spaces described in terms of Riemannian symmetric pairs (G, K).

Rank and Sectional Curvature

The rank of is at least one with equality if the sectional curvature is positive or negative. If the sectional curvature is positive the space is of compact type, and if the sectional curvature is negative the space is of noncompact type.

The rank of a Euclidean type space is equal to its dimension. This implies Euclidean type spaces are isometric to Euclidean space of that dimension.

Therefore we are reduces to classifying symmetric spaces of compact and noncompact type.

In both cases, we have two classes of symmetric spaces described in terms of Riemannian symmetric pairs (G, K).

Rank and Sectional Curvature

The rank of is at least one with equality if the sectional curvature is positive or negative. If the sectional curvature is positive the space is of compact type, and if the sectional curvature is negative the space is of noncompact type.
The rank of a Euclidean type space is equal to its dimension. This implies Euclidean type spaces are isometric to Euclidean space of that dimension.
Therefore we are reduces to classifying symmetric spaces of compact and noncompact type.
In both cases, we have two classes of symmetric spaces described in terms of Riemannian symmetric pairs (G, K).

Classification

Compact type:

- $G = H \times H$ where H is a simply connected compact Lie group and K is the diagonal subgroup.
- G is the complexification of a simply connected noncompact simple Lie group and K is the maximal compact subgroup.

Noncompact type:

- G is a simply connected complex simple Lie group and K is the maximal compact subgroup.
- G is a simply connected noncompact simple Lie group and K is the maximal compact subgroup.

By the classification of Lie groups all such symmetric spaces are also classified, and this finishes Cartan's classification.

Classification

Compact type:

- $G = H \times H$ where H is a simply connected compact Lie group and K is the diagonal subgroup.
- G is the complexification of a simply connected noncompact simple Lie group and K is the maximal compact subgroup.

Noncompact type:

- G is a simply connected complex simple Lie group and K is the maximal compact subgroup.
- G is a simply connected noncompact simple Lie group and K is the maximal compact subgroup.

By the classification of Lie groups all such symmetric spaces are also classified, and this finishes Cartan's classification.

Classification

Compact type:

- $G = H \times H$ where H is a simply connected compact Lie group and K is the diagonal subgroup.
- G is the complexification of a simply connected noncompact simple Lie group and K is the maximal compact subgroup.

Noncompact type:

- G is a simply connected complex simple Lie group and K is the maximal compact subgroup.
- G is a simply connected noncompact simple Lie group and K is the maximal compact subgroup.

By the classification of Lie groups all such symmetric spaces are also classified, and this finishes Cartan's classification.

Outline

References

Thanks!

- J. Eschenburg: Lecture Notes on Symmetric Spaces, University of Augsburg.
- X. Gao: Symmetric Spaces, University of Illinois, 2014.
- P. Holmelin: Symmetric Spaces, Master's Thesis, Lund Institute of Technology, 2005.
- P. Petersen: Riemannian Geometry Third Edition, Springer, AG, Switzerland, 2016.