The functional equation for Riemann zeta over function fields

Henry Twiss
University of Minnesota

April 2019

Contents

1 Statement and Proof of the Theorem

Using the Riemann-Roch theorem we're going to prove the functional equation for the ζ-function associated to a function field of transcendence degree one over $\mathbb{F}_q$. Recall that we define said ζ-function by

$$\zeta_K(s) = \sum_{A \geq 0} \frac{1}{(NA)^s},$$

where NA is the norm of A (precisely it's $q^{-\deg(A)}$) As a consequence we will show that the ζ-function satisfies a rational function in $u = q^{-s}$.

Theorem 1.1 (The Functional Equation)**.** Let K be a function field of transcendence degree one over $\mathbb{F}_q$, and let ζ_K be the corresponding ζ-function. Define $\xi(s) = q^{(g-1)s}\zeta_K(s)$ where g is the genus of K. Then $\xi(s) = \xi(1-s)$.

Proof. Let $\deg(A) = n$, so that $NA^{-s} = q^{-ns}$. Summing over $n \geq 0$ and making the substitution $u = q^{-s}$ we may write

$$\zeta_K(s) = \sum_{n \geq 0} b_n u^n := Z_K(u) \quad \text{and} \quad \zeta_K(1-s) = \sum_{n \geq 0} b_n \left(\frac{1}{qu}\right)^n := Z_K\left(\frac{1}{qu}\right),$$

where we define b_n to be the number of effective divisors $A \in \mathcal{D}_K$ of degree n (a divisor A is effective if $A \geq 0$). The functional equation now takes the form

$$u^{1-g} Z_K(u) = \left(\frac{1}{qu}\right)^{1-g} Z_K\left(\frac{1}{qu}\right).$$

Now notice if we substitute $\frac{1}{qu}$ for u on the left-hand side we get the right-hand side. Denoting the left-hand side as $F'(u)$, the theorem is equivalent to showing $F'(u) = F'(\frac{1}{qu})$. To make the computations easier to work with we're going to show the equivalent statement $F(u) = F(\frac{1}{qu})$ where $F(u) = (q-1)F'(u)$. We will accomplish this by rewriting the sum on the left-hand side as a sum over divisor classes and using Riemann-Roch. Define Cl_K^+ to be the set of all divisor classes $\overline{A}$ such that $\deg(\overline{A}) \geq 0$ (we can do this because all the divisors in a class have the same degree) and let Cl_K^i be the set of all divisor classes $\overline{A}$ such that $0 \leq \deg(\overline{A}) \leq i$.

$$
\begin{aligned}
F(u) &= (q-1)u^{1-g}\sum_{n\geq 0} b_n u^n & (1)\\
&= (q-1)u^{1-g}\left(\sum_{\overline{A}\in Cl_K^+} \frac{q^{l(\overline{A})-1}}{q-1} u^{\deg(\overline{A})}\right) & (2)\\
&= u^{1-g}\left(\sum_{\overline{A}\in Cl_K^+} q^{l(\overline{A})}u^{\deg(\overline{A})} - \sum_{\overline{A}\in Cl_K^+} u^{\deg(\overline{A})}\right) & (3)\\
&= u^{1-g}\left(\sum_{\overline{A}\in Cl_K^{2g-2}} q^{l(\overline{A})}u^{\deg(\overline{A})} + \sum_{\{\overline{A}\mid \deg(\overline{A})\geq 2g-1\}} q^{l(\overline{A})}u^{\deg(\overline{A})} - \sum_{\overline{A}\in Cl_K^+} u^{\deg(\overline{A})}\right) & (4)\\
&= \sum_{\overline{A}\in Cl_K^{2g-2}} q^{l(\overline{A})}u^{\deg(\overline{A})-g+1} + h_k\left(\sum_{n=2g-1}^{\infty} q^{n-g+1}u^{n-g+1} - \sum_{n=0}^{\infty} u^{n-g+1}\right) & (5)\\
&= \sum_{\overline{A}\in Cl_K^{2g-2}} q^{l(\overline{A})}u^{\deg(\overline{A})-g+1} + h_k\left(\frac{q^g u^q}{1-qu} - \frac{u^{1-g}}{1-u}\right) & (6)
\end{aligned}
$$

- $1 \to 2$: Sum over effective divisors instead of degree. Also, we have a theorem which says the number of effective divisors in $\overline{A}$ is exactly $\frac{q^{l(\overline{A})-1}}{q-1}$.
- $2 \to 3$: Use the $q-1$ term to clear denominators.
- $3 \to 4$: Break up the first sum by degree $2g-2$ (Riemann-Roch).
- $4 \to 5$: This follows since there are always $h_K = |Cl_K^0|$ classes of degree n (h_K is independent of n), and a corollary of Riemann-Roch which says that if $\deg(\overline{A}) \geq 2g-1$ then $l(\overline{A}) = n-g+1$ where $\deg(\overline{A}) = n$.
- $5 \to 6$: Reindex the middle sum to start from $n=0$ and notice the two latter sums are infinite geometric series which converge because $u = q^{-s}$ with $\Re(s) > 1$.

A direct substitution $u \mapsto \frac{1}{qu}$ will show that the second term remains unchanged. In fact, the two terms in the difference switch under this substitution. So, all that remains to show is the summation remains unchanged. Now let $\overline{C} \in Cl_K$ be the canonical class. By a corollary of Riemann-Roch, $\deg(\overline{C}) = 2g-2$ so that the map $\varphi : Cl_K^{2g-2} \to Cl_K^{2g-2}$ given by $\overline{A} \mapsto \overline{C} - \overline{A}$ is a well-defined involution. So in particular, is a bijection. Then

$$\sum_{\overline{A} \in Cl_K^{2g-2}} q^{l(\overline{A})} u^{\deg(\overline{A})-g+1} = \sum_{\overline{A} \in Cl_K^{2g-2}} q^{l(\overline{C}-\overline{A})} u^{\deg(\overline{C}-\overline{A})-g+1} \tag{7}$$

$$= \sum_{\overline{A} \in Cl_K^{2g-2}} q^{l(\overline{A})-\deg(\overline{A})+g-1} u^{-\deg(\overline{A})+g-1}. \tag{8}$$

- $7_{LHS} \to 7_{RHS}$: Note φ is a bijection.
- $7 \to 8$: Use Riemann-Roch thrice. The first two times use it for $\overline{A} - \overline{C}$ and $\overline{A}$, and combine results. This gives the change in power for u. The change in power for q is just Riemann-Roch again applied to $\overline{A}$. .

On the other hand, if we make the substitution $u \mapsto \frac{1}{qu}$:

$$\sum_{\overline{A} \in Cl_K^{2g-2}} q^{l(\overline{A})} \left(\frac{1}{qu}\right)^{\deg(\overline{A})-g+1} = \sum_{\overline{A} \in Cl_K^{2g-2}} q^{l(\overline{A})-\deg(\overline{A})+g-1} u^{-\deg \overline{A}+g-1},$$

we get the same result as above. This proves $F(u) = F(\frac{1}{qu})$ as desired. $\square$

2 Some Final Comments

Having proven the functional equation we now want to show the existence of a polynomial $L_K(u)$ of degree $2g$ such that

$$\zeta_K(s) = \frac{L_K(u)}{(1-u)(1-qu)}.$$

As before, take $\zeta_K(s) = Z_K(u)$. It's a consequence of Riemann-Roch that if $n > 2g-2$, then $b_n = h_K \frac{q^{n-g+1}-1}{q-1}$. If we substitute this into $Z_K(u)$ and sum the geometric series for terms with $n > 2g-2$ we find

$$Z_K(u) = \sum_{n=0}^{2g-2} b_n u^n + \frac{h_K}{q-1}\left(\frac{q^g}{1-qu} - \frac{1}{1-u}\right) u^{2g-1}.$$

If we combine the terms inside the parenthesis then their numerator divides $q-1$ since $q^g(1-u) - (1-qu) = (q^g-1) - qu(q^{g-1}-1)$ (we can manually check the case $g=0$). Collecting all terms under the common denominator gives the result. We now wish to find the degree of $L_K(u)$. Start by noticing $L_K(0) = 1$ because $b_0 = 1$. Since $u^{1-g}Z_K(u)$ is invariant under $u \mapsto \frac{1}{qu}$, computing $u^{1-g}Z_K(u)$ and using the substitution $u \mapsto \frac{1}{qu}$ easily shows $q^{-g}u^{-2g}L_K(u) = L_K(\frac{1}{qu})$. If we take the limit as $u \to \infty$, then $L_K(\frac{1}{qu}) \to 1$ and hence $q^{-g}u^{-2g}L_K(u) \to 1$. This implies $L_K(u)$ is of degree $2g$ and in fact its leading term is $q^g u^{2g}$. This rational expression for the ζ-function has some other interesting properties such as giving an analytic continuation to all of $\mathbb{C}$ with simple poles at $s=0$ and $s=1$. Also, $L_K'(0) = a_1 - 1 - q$ and $L_K(1) = h_K$ where a_1 is the number of primes of K of degree one. These facts are easy to check and we omit their proof. On that note we are finished.