The root datum of special linear and projective linear groups

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1 Motivation

In the following all linear algebraic groups are assumed semisimple. Linear algebraic groups are of crucial importance in the study of algebraic geometry. We would like to understand the structure of these linear algebraic groups, and studying maps to and from the group can be more fruitful than working with the group directly. Characters and cocharacters are these maps and give rise to root systems. Root systems are the primary invariant for linear algebraic groups, but they do not completely characterize the linear algebraic group. We would like to extend this invariant to completely characterize the linear algebraic group; this is the root datum. Two prototypical examples of linear algebraic groups are

$$\mathrm{SL}_2(\mathbb{C}) = \{u \in \mathrm{GL}_2(\mathbb{C}) \mid \det u = 1\} \quad \text{and} \quad \mathrm{PGL}_2(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C})/Z(\mathrm{GL}_2(\mathbb{C})).$$

The root datum of these two linear algebraic groups are $(\mathbb{Z}, \Phi, \mathbb{Z}, \Phi^\vee)$ and $(\mathbb{Z}, \Phi, \mathbb{Z}, \Phi^\vee)$; so how does the root datum really distinguish between these two linear algebraic groups? It's more subtle than one might think.

2 Characters and Cocharacters

The setting is a linear algebraic group G over $\mathbb{C}$. On G we have maps $\chi : G \to \mathbb{C}$ called characters which are multiplicative homomorphisms. We denote the set of characters on G by $X(G)$. Dual to the characters, we have cocharacters: multiplicative homomorphisms $\chi^\vee : \mathbb{C} \to G$. We denote the set of cocharacters on G by $X^\vee(G)$. The characters and cocharacters form respective groups under multiplication.

In G there exists subgroups T called tori which are characterized by being isomorphic to the set of diagonal matrices D_k in $\mathrm{GL}_k(\mathbb{C})$ for $k \leq n$. With respect to inclusion, there exists maximal tori T (we may have more than one but they all isomorphic up to conjugation). Choosing a maximal torus T, we may restrict the characters and cocharacters to T. We usually consider characters on T, and write $X(T)$ and $X^\vee(T)$ for the set of characters and cocharacters restricted to T respectively.

3 Characters and Cocharacters of $\mathbf{SL}_2(\mathbb{C})$ and $\mathbf{PGL}_2(\mathbb{C})$

Let's determine the characters and cocharacters with respect to a maximal torus for $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{PGL}_2(\mathbb{C})$:

- A maximal torus T for $\mathrm{SL}_2(\mathbb{C})$ is given by the set of matrices

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \qquad (t \in \mathbb{C}).$$

  Any character $\chi : T \to \mathbb{C}$ must be a multiplicative homomorphism, so the characters take the form

$$\chi \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = t^n \qquad (n \in \mathbb{Z}).$$

  Dually, the cocharacters $\chi^\vee : \mathbb{C} \to T$ are given by the maps

$$\chi^\vee(t) = \begin{pmatrix} t^n & 0 \\ 0 & t^{-n} \end{pmatrix} \qquad (n \in \mathbb{Z}).$$

  In particular, we can identify the character and cocharacter groups with $\mathbb{Z}$.

- A maximal torus T for $\mathrm{PGL}_2(\mathbb{C})$ is given by the classes

$$\overline{\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}} \qquad (t \in \mathbb{C}).$$

  Analogous to the case for $\mathrm{SL}_2(\mathbb{C})$, the characters are

$$\chi \overline{\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}} = t^n \qquad (t \in \mathbb{C}),$$

  and the cocharacters are

$$\chi^\vee(t) = \overline{\begin{pmatrix} t^n & 0 \\ 0 & 1 \end{pmatrix}}.$$

  In particular, we can identify the characters with $\mathbb{Z}$ and the cocharacters with $\mathbb{Z}$.

4 Root Systems

Defined the root system Φ of G by

$$\Phi := \{\chi \in X(T) \mid \chi \neq 0,\, \mathfrak{g}_\chi \neq 0\},$$

where $\mathfrak{g}_\chi \neq 0$ is a technical condition which means there exists some matrix u such that for any $v \in T$ we have $v^{-1}uv = \chi(v)u$. In other words, $\chi|_T$ acts like conjugation by T for at least one matrix u. We usually label roots by α. Dual to every root α, there is a unique coroot $\alpha^\vee \in X^\vee(T)$ defined by $\langle \alpha, \alpha^\vee \rangle = 2$. These coroots form a coroot system $\Phi^\vee$ not necessarily isomorphic to Φ.

5 Root and Coroot Systems of $\mathbf{SL}_2(\mathbb{C})$ and $\mathbf{PGL}_2(\mathbb{C})$

Both $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{PGL}_2(\mathbb{C})$ have root systems of type A_1 meaning there is only one root (and its negative). Root systems of type A_1 are self-dual so the coroot system also has one root (and its negative). Let's describe the root and coroot for $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{PGL}_2(\mathbb{C})$.

- The root α for $\mathrm{SL}_2(\mathbb{C})$ is the map

$$\alpha \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} = t^2.$$

  The corresponding coroot $\alpha^\vee$ is the map

$$\alpha^\vee(t) = \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}.$$

- The root α for $\mathrm{PGL}_2(\mathbb{C})$ is the map

$$\alpha \overline{\begin{pmatrix} t & \\ & 1 \end{pmatrix}} = t.$$

  The corresponding coroot $\alpha^\vee$ is the map

$$\alpha^\vee(t) = \overline{\begin{pmatrix} t^2 & \\ & 1 \end{pmatrix}}.$$

6 Lattices

The root lattice of Φ is the $\mathbb{Z}$-module generated by Φ. Dually, the coroot lattice is the $\mathbb{Z}$-module generated by $\Phi^\vee$. Denoting the lattices by $\mathbb{Z}\Phi$ and $\mathbb{Z}\Phi^\vee$, its a general fact that $\mathbb{Z}\Phi \subset X(T)$ and $\mathbb{Z}\Phi^\vee \subset X^\vee(T)$ and these subgroup are invariants of G. This additional combinatorical data with the root datum classifies the semisimple linear algebraic group completely, and gives us a way to distinguish between $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{PGL}_2(\mathbb{C})$ purely from the root information.

7 Root Datum

The root datum of G is the quadruple $(X(T), \Phi, X^\vee(T), \Phi^\vee)$ with an implicit bijection $\Phi \to \Phi^\vee$ (this is the map $\alpha \to \alpha^\vee$). The root dataum classifies the semisimple linear algebraic group completely by a theorem of Chevalley:

Theorem 7.1 (Chevalley Classification Theorem)**.** Two semisimple linear algebraic groups are isomorphic if and only if they have isomorphic root datum. To each root datum there exists a semisimple linear algebraic group which realizes it.

The importance piece to take away from this theorem is that with the root datum we can deduce how the root and coroot lattices fit inside the space of characters, and its this data which characterizes the semisimple linear algebraic group.

8 Root Datum of $\mathbf{SL}_2(\mathbb{C})$ and $\mathbf{PGL}_2(\mathbb{C})$

Let's show $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{PGL}_2(\mathbb{C})$ and not isomorphic by only using the root datum. By the Chevalley classification theorem we only need to show the root datum are not isomorphic.

- The root datum of $\mathrm{SL}_2(\mathbb{C})$ is $(\mathbb{Z}, \Phi, \mathbb{Z}, \Phi^\vee)$. The root α is

$$\alpha \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} = t^2$$

so any element $n\alpha$ in $\mathbb{Z}\Phi$ is of the form

$$n\alpha \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} = t^{2n}.$$

Therefore $\mathbb{Z}\Phi$ is an index 2 subgroup of $\mathbb{Z}$. The coroot is

$$\alpha^\vee(t) = \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}$$

so any element $n\alpha^\vee$ in $\mathbb{Z}\Phi^\vee$ is of the form

$$n\alpha^\vee(t) = \begin{pmatrix} t^n & \\ & t^{-n} \end{pmatrix}.$$

Therefore $\mathbb{Z}\Phi^\vee$ the whole $\mathbb{Z}$.

- The root datum of $\mathrm{PGL}_2(\mathbb{C})$ is $(\mathbb{Z}, \Phi, \mathbb{Z}, \Phi^\vee)$. The root α is

$$\alpha \overline{\begin{pmatrix} t & \\ & 1 \end{pmatrix}} = t$$

so any element $n\alpha$ in $\mathbb{Z}\Phi$ is of the form

$$n\alpha \overline{\begin{pmatrix} t & \\ & 1 \end{pmatrix}} = t^n.$$

Therefore $\mathbb{Z}\Phi$ is the whole $\mathbb{Z}$. The coroot is

$$\alpha^\vee(t) = \overline{\begin{pmatrix} t^2 & \\ & 1 \end{pmatrix}}$$

so any element $n\alpha^\vee$ in $\mathbb{Z}\Phi^\vee$ is of the form

$$n\alpha^\vee(t) = \overline{\begin{pmatrix} t^{2n} & \\ & 1 \end{pmatrix}}.$$

Therefore $\mathbb{Z}\Phi^\vee$ is an index 2 subgroup of $\mathbb{Z}$.

This tells us that the roots and coroots of $\mathrm{SL}_2(\mathbb{C})$ fit into the corresponding lattices dual to how the root and coroots of $\mathrm{PGL}_2(\mathbb{C})$ fit into the corresponding lattices. Because of this reason the root datum cannot be isomorphic and so $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{PGL}_2(\mathbb{C})$ are not isomorphic.