Using holonomy to understand Riemannian manifolds

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1 Motivation and History

Let us start with an example to illustrate the idea of holonomy. Recall that parallel translation preserves inner products so it preserves length of vectors and angles as well. Let S^2 be the unit sphere, γ_1 and γ_2 be two meridians through the north pole

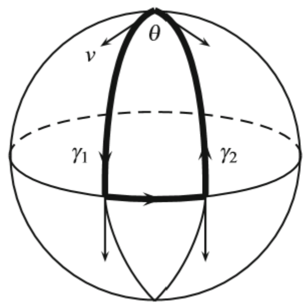

Figure 1: Parallel translating v along a closed loop.

p making an angle θ with each other, and let v be a vector at p tangent to γ_1 (see Figure 1). Since γ_1 is a geodesic its tangent vector field is parallel, so the parallel translate of v to the equator remains parallel to γ_1 and therefore remains parallel to the equator at all times. When it reaches γ_2 along the equator it will be parallel to γ_2. Parallel translating v along γ_2 results in a tangent vector at p, but this is not the same vector as v. The phenomena that geometric information can be lost by parallel translating around a closed loop is *holonomy*. We can also think of holonomy as measuring the extent to which parallel transport around a loop fails to preserve the geometric information being transported.

In 1926 an idea came about from Élie Cartan to help classify symmetric spaces. This idea was holonomy. Cartan considered the Levi-Civita connection of a Riemannian manifold M, so that the holonomy group was a Lie subgroup of the orthogonal group (in the modern language this is Riemannian holonomy). In the end, Cartan used Holonomy to completely classify symmetric spaces. Studying holonomy in the more general setting lead to two crucial theorems: one of de Rham and one of Ambrose-Singer. The former provides a description of a Riemannian manifold locally as a product of Riemannian manifolds by splitting the tangent bundle locally under the action of the holonomy. The later establishes a close relationship between the holonomy of a connection and the curvature of that connection. Berger was then able to use the Ambrose-Singer theorem to classify the holonomy of a Riemannian manifold. It was only recently that each holonomy in Berger's classification is realized by a Riemannian manifold. Time permitting we will discuss these two crucial theorems and Berger's classification.

2 The Holonomy Group

Let M be a connected Riemannian manifold with its Levi-Civita connection and let $\gamma : [0, 1] \to M$ be piecewise smooth curve with initial point p and end point q. By the general theory of ODEs there exists a unique piecewise smooth parallel vector field $V(t)$ along the curve given any initial vector $V(t) \in T_pM$ (this parallel vector field is dependent on the connection ∇ because the covariant derivative is dependent on ∇). For every $V(t) \in T_pM$, parallel translation along γ defines a parallel transport map

$$P_\gamma : T_pM \to T_qM \qquad V(0) \mapsto V(1).$$

If γ is a loop based at p, this map is a linear isomorphism of T_pM since parallel translation respects vector addition and scalar multiplication and γ has an inverse $-\gamma$. Considering all piecewise smooth loops at p, and observing that the construction of P_γ depends only on p, ∇, and γ, we define the holonomy group of ∇ at p to be

$$\mathrm{Hol}_p(\nabla) := \{P_\gamma \in \mathrm{GL}(\mathbb{R}^n) \mid \gamma \text{ a piecewise smooth loop at } p\}.$$

Often we work with the restricted holonomy group $\mathrm{Hol}^0_p(\nabla)$ which is defined exactly the same as the regular holonomy group except that γ is required to be contractible. The following list is some basic facts about holonomy which we will use in the following but not prove:

- $\mathrm{Hol}_p(\nabla) \subset \mathrm{O}(T_pM) = \mathrm{O}(n)$.
- $\mathrm{Hol}^0_p(\nabla)$ is a connected normal subgroup of $\mathrm{Hol}_p(\nabla)$.
- $\mathrm{Hol}_p(\nabla) \subset \mathrm{SO}(n)$ if and only if M is orientable.
- $\mathrm{Hol}_p(\nabla)$ and $\mathrm{Hol}^0_p(\nabla)$ are Lie groups, the latter is compact while the former is not.

Let's compute some holonomy groups:

- Consider $\mathbb{R}^n$ with the flat metric. Then parallel translation reduces to translation in $\mathbb{R}^n$ which implies for any γ that the induced map P_γ is the identity (since γ is closed). This means $\mathrm{Hol}_p(\nabla) = \{1\}$ is trivial. Since restricted holonomy is a subgroup of holonomy, the restricted holonomy group is also trivial.
- Let S^2 be the unit 2-sphere with standard metric and Levi-Civita connection. We claim $\mathrm{Hol}_p(S^2) = \mathrm{SO}(2)$ for all p. Fix p and declare it to be the north pole. The forward inclusion has already been discussed. For any $\theta \in \mathrm{SO}(2)$ consider

the path γ sending p to the equator along any geodesic, along the equator, and then back along the geodesic to the pole making an angle θ with the geodesic leaving the pole. As we parallel transport any tangent vector along this loop the angle with the velocity field of the geodesic is preserved so the induced parallel transport map is rotation by θ. In fact, is the example we started with.

3 de Rham's Decomposition Theorem

The de Rham decomposition theorem says a Riemannian manifold can locally be decomposed as a product of Riemannian manifolds according to the holonomy of the Levi-Civita connection.

To be more precise, first consider the the following facts:

- If M and N are Riemannian manifolds with Levi-Citiva connections ∇_M and ∇_N respectively, $\mathrm{Hol}_{(p,q)}(\nabla_M \times \nabla_N) = \mathrm{Hol}_p(\nabla_M) \times \mathrm{Hol}_p(\nabla_N)$.
- $\mathrm{Hol}_p(\nabla)$ is defined up to conjugation: $\mathrm{Hol}_q(\nabla) = P_\gamma \mathrm{Hol}_p(\nabla) P_\gamma^{-1}$ where γ is a piecewise smooth path from p to q.

This tells us the holonomy of a product of Riemannian manifolds decomposes into the product of the individual holonomies. Since the holonomy is defined up to conjugation; if the holonomy decomposes at one points then it decomposes everywhere. Fixing $p \in M$, we have a natural action of $\mathrm{Hol}_p^0(\nabla)$ on T_pM defined by

$$\mathrm{Hol}_p^0(\nabla) \times T_pM \to T_pM \qquad (P_\gamma, v) \mapsto P_\gamma(v).$$

If E is a $\mathrm{Hol}_p^0(\nabla)$-invariant subspace of T_pM, then we may write

$$T_pM = E_1 \oplus \cdots \oplus E_k$$

where the E_i are irreducible $\mathrm{Hol}_p^0(\nabla)$-invariant subspaces. Parallel translation along loops and more generally paths preserves this splitting by construction, so we get a splitting of the tangent bundle TM:

$$TM = \eta_1 \oplus \cdots \oplus \eta_k$$

into subspaces invariant under parallel translation. de Rham's theroem is the following:

Theorem 3.1 (de Rham's Decomposition Theorem)**.** Let M be a Riemannian manifolds with its Levi-Civita connection, and consider the natural splitting

$$TM = \eta_1 \oplus \cdots \oplus \eta_k.$$

Then for each $p \in M$, there exists a neighborhood U such that

$$(U, \nabla|_U) = (U_1 \times \cdots \times U_k, \nabla|_{U_1} \times \cdots \times \nabla|_{U_k})$$

with $TU_i = \eta_i|_{U_i}$ for $1 \le i \le k$.

In other words, every Riemannian manifold locally looks like a product of Riemannian manifolds (with the Levi-Citiva connection) where the product is determined by the holonomy. The main takeaway is that it usually suffices to study the holonomy of manifolds whose tangent spaces (or bundles) are irreducible with respect to the holonomy.

4 Ambrose-Singer Theorem

The curvature and the holonomy of a Riemannian manifold with the Levi-Citiva connection are strictly related. One can construct a family of endomorphisms of T_pM using the curvature tensor and parallel translation. The linear subspace generated by these maps is precisely $\mathfrak{hol}_p(\nabla)$ (the Lie algebra of $\mathrm{Hol}_p(\nabla)$). This idea is captured in the Ambrose-Singer theorem. The theorem is the following:

Theorem 4.1 (Ambrose-Singer Holonomy Theorem)**.** Let M be a Riemannian manifold with ∇ the Levi-Civita connection. Fix $p \in M$, so $\mathfrak{hol}_p(\nabla)$ is a Lie subalgebra of $\mathrm{End}(T_pM)$. Then $\mathfrak{hol}_p\nabla$ is the subalgebra generated by the endomorphisms $P_\gamma^{-1} \circ R(P_\gamma u, P_\gamma v) \circ P_\gamma$ (here $R(P_\gamma u, P_\gamma v)$ is the Riemannian curvature tensor at T_pM) where γ is a piecewise smooth curve from p to q and $u, v \in T_pM$.

The takeaway here is that infinitesimal holonomy at a point is the curvature of the manifold at that point. This isn't quite clear from the statement of the theorem, so let's take a look at the sketch of the proof. The idea is capitulated in Figure 2.

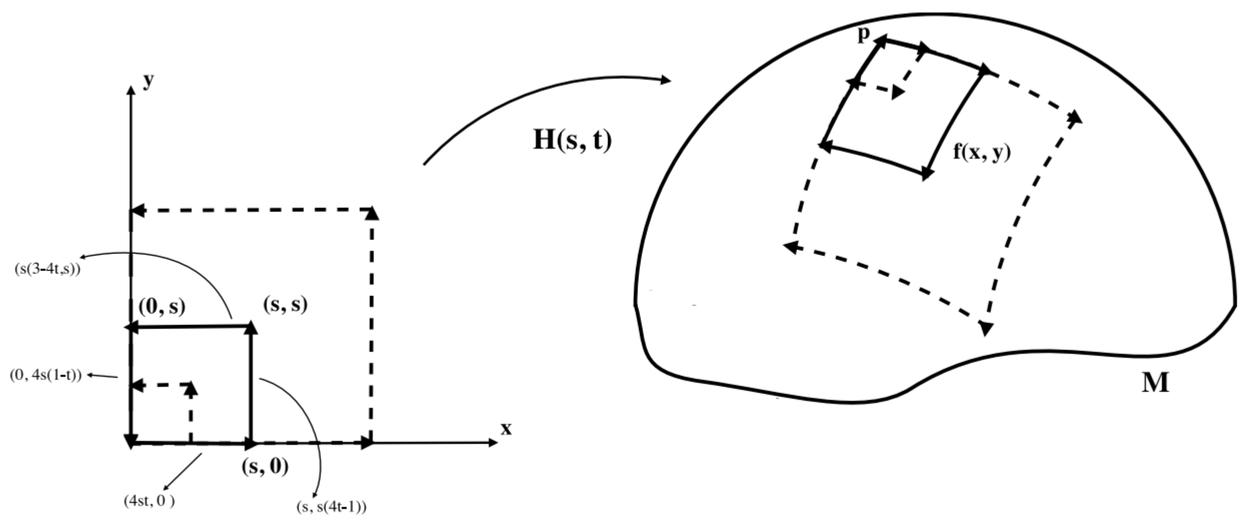

Figure 2: The shrinking homotopy and its image in M.

Proof Sketch. One considers a small shrinking rectangle in M given by the homotopy:

$$H(s,t) = \begin{cases} f(4st, 0) & 0 \le t \le \frac{1}{4} \\ f(s, s(4t-1)) & \frac{1}{4} \le t \le \frac{1}{2} \\ f(s(3-4t), s) & \frac{1}{2} \le t \le \frac{3}{4} \\ f(0, 4s(1-t)) & \frac{3}{4} \le t \le 1 \end{cases}.$$

Here $f : U \to M$ is a smooth map used to move the homotopy in $\mathbb{R}^2$ into M (see Figure 2).

Letting γ be a piecewise smooth path from p to q we define $\gamma_s = \gamma h_s \gamma^{-1}$. We then get a family of parallel transport maps P_s such that $P_s : [0,1] \to \mathrm{GL}(T_pM)$ is a smooth curve in $\mathrm{Hol}_p^0(\nabla)$, the derivative of this curve at $s = 0$ is the change in the parallel transport map for an infinitesimally small rectangle and the derivative is precisely $P_\gamma^{-1} \circ R(P_\gamma u, P_\gamma v) \circ P_\gamma$. □

5 Reducible Manifolds and Symmetric Spaces

Before we discuss Berger's classification we need a small discussion about reducible manifolds and symmetric spaces. We need these facts to discuss Berger's list, but they can be removed at the expense of making the list longer.

We call a Riemannian manifold M locally reducible if every point $p \in M$ has an open neighborhood isomorphic to a Riemannian product $(P \times Q, \nabla_P \times \nabla_Q)$. We say M is irreducible if it is not locally reducible.

A Riemannian manifold M is said to be a symmetric space if for every point $p \in M$ there exists an isometric involution $s_p : M \to M$, such that p is an isolated fixed point of s_p. Some prototypical examples of symmetric spaces are $\mathbb{R}^n$ and S^n with the usual metrics. M is said to be locally symmetric if every point has a neighborhood isometric to a symmetric space. We say M is nonsymmetric if it is not locally symmetric. It's a surprising fact that a Riemannian manifold with the Levi-Civita connection is locally symmetric if and only if it has constant curvature.

6 Berger's Classification

In 1955 Berger proved his classification theorem:

Theorem 6.1 (Berger Classification Theorem)**.** Suppose M is an n-dimensional Riemannian manifold with the Levi-Civita connection satisfying the following properties:

(i) M is simply-connected.

(ii) M is irreducible.

(iii) M is nonsymmetric.

Then exactly one of the following cases hold for all $p \in M$:

- $\mathrm{Hol}_p(\nabla) = \mathrm{SO}(n)$.
- $n = 2m$ with $m \geq 2$, and $\mathrm{Hol}_p(\nabla) = \mathrm{U}(m)$ in $\mathrm{SO}(2m)$.
- $n = 2m$ with $m \geq 2$, and $\mathrm{Hol}_p(\nabla) = \mathrm{SU}(m)$ in $\mathrm{SO}(2m)$.
- $n = 4m$ with $m \geq 2$, and $\mathrm{Hol}_p(\nabla) = \mathrm{Sp}(m)$ in $\mathrm{SO}(4m)$.
- $n = 4m$ with $m \geq 2$, and $\mathrm{Hol}_p(\nabla) = \mathrm{Sp}(m)\mathrm{Sp}(1)$ in $\mathrm{SO}(4m)$.
- $n = 7$, and $\mathrm{Hol}_p(\nabla) = G_2$ in $\mathrm{SO}(7)$.
- $n = 8$, and $\mathrm{Hol}_p(\nabla) = \mathrm{Spin}(7)$ in $\mathrm{SO}(8)$.

The idea behind Berger's theorem is to apply two tests to arbitrary groups to see if they could be holonomy groups. Let's sketch the proof of Berger's theorem:

Proof Sketch. Under the assumptions, $\mathrm{Hol}_p(\nabla)$ is a closed connected Lie subgroup of $\mathrm{SO}(n)$. Let $H := \mathrm{Hol}_p(\nabla)$ be an arbitrary group and $\mathfrak{h}$ be its Lie algebra. We apply two tests to H that all holonomy groups must satisfy:

- If S is a constant tensor ($\nabla S = 0$), then the Riemannian curvature tensor R_{abcd} lies in the vector subspace $S^2\mathfrak{h}$. If $\mathfrak{h}$ has large codimension in $\mathfrak{so}(n)$, then the vector subspace, $\mathfrak{R}^H$, of $S^2\mathfrak{h}$ satisfying the first Bianchi identity will be small, but by Ambrose-Singer $\mathfrak{R}^H$ needs to be large enough to generate $\mathfrak{h}$. Most H fail this condition. This is the first test.

- The tensor $\nabla_e R_{abcd}$ lies in $(\mathbb{R}^n)^* \otimes \mathfrak{R}^H$ and satisfies the second Bianchi identity. Often this implies R is constant ($\nabla R = 0$) so g is locally symmetric, and therefore we need to exclude such H by the assumptions. This is the second test.

□

All three conditions (i), (ii), and (iii) can be removed at the expense of making Berger's list longer. If (i) is removed we need to include non-connected Lie groups whose identity components are already on the list. If (ii) is removed we need to include all products of groups on the list. If (iii) is removed we need to include holonomy groups of symmetric spaces which are known from Cartan's classification.

When Berger proved this theorem he knew these were the only possibilities for the holonomy group, but he didn't know each possibility was realized by a Riemannian manifold. Around 1895 it was shown that each group on Berger's list was realized by a Riemannian manifold. We also characterize these groups by the following:

- $\mathrm{SO}(n)$ is the holonomy of generic metrics.
- Metrics on manifolds with $\mathrm{Hol}_p(\nabla) = \mathrm{U}(m)$ are called Kähler metrics.
- Metrics on manifolds with $\mathrm{Hol}_p(\nabla) = \mathrm{SU}(m)$ are called Calabi-Yau metrics. In particular, all Calabi-Yau metrics are Kähler.
- Metrics on manifolds with $\mathrm{Hol}_p(\nabla) = \mathrm{Sp}(m)$ are called hyperkähler metrics. They are Ricci-flat and Kähler.
- Metrics on manifolds with $\mathrm{Hol}_p(\nabla) = \mathrm{Sp}(m)\mathrm{Sp}(1)$ are called quaternionic Kähler metrics. They are not Kälher, nor Ricci-flat, but are Einstein.
- The holonomy groups G_2 and $\mathrm{Spin}(7)$ are called the exceptional holonomy groups.

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