Notes: Differentials Forms in Algebraic Topology - Bott & Tu

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1 The de Rham Complex on $\mathbb{R}^n$

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- > ... So let $x_1, \ldots, x_n$ be the linear coordinates on $\mathbb{R}^n$ ...

  The linear coordinates $x_1, \ldots, x_n$ are the coordinate functions of the identity map on $\mathbb{R}^n$. In other words, $x_i(y) = y_i$ where y_i is the i-th coordinate of $y \in \mathbb{R}^n$. If you are familiar with manifolds, then the x_i's are the coordinate functions of the trivialization id for the atlas $\{\text{id}, \mathbb{R}^n\}$ for $\mathbb{R}^n$.

- > ... We define Ω^* to be the algebra over $\mathbb{R}$ generated by the $dx_1, \ldots, dx_n$ with the relations ...

  The dx_i's are the dual maps of the basis vectors $\partial/\partial x_i$'s where $\partial/\partial x_i$ is the linear operator which sends a function $f : \mathbb{R}^n \to \mathbb{R}$ to its partial derivative with respect to x_i, that is $\partial f/\partial x_i$. Here the basis vectors $\partial/\partial x_i$ form a natural vector space over $\mathbb{R}$ which acts on functions f in the obvious way. We will call the dx_i's differentials when we refer to them in these notes.

- > $$\Omega^*(\mathbb{R}^n) = \{\mathbb{C}^\infty \text{ functions on } \mathbb{R}^n\} \otimes_{\mathbb{R}} \Omega^*$$

  The C^∞ functions on $\mathbb{R}^n$ are functions that take values in $\mathbb{R}$.

- > ... We also write $\omega = \sum f_I \, dx_I$ ...

  The indexing set I is a subset of $\{1, 2, \ldots, n\}$ and we sum over a finite amount of such indexing sets.

- > ... The algebra $\Omega^*(\mathbb{R}^n) = \bigoplus_{q=0}^n \Omega^q(\mathbb{R}^n)$ is naturally graded, where $\bigoplus_{q=0}^n \Omega^q(\mathbb{R}^n)$ consists of the C^∞ q-forms on $\mathbb{R}^n$ ...

  q-forms ω are forms where $|I| = q$ for all I in $\omega = \sum f_I \, dx_I$.

- i) if $f \in \Omega^0(\mathbb{R}^n)$, then $df = \sum \partial f/\partial x_i \, dx_i$
  ii) if $\omega = \sum f_I \, dx_I$, then $d\omega = \sum df_I \, dx_I$

  The definition given for the differential operator d is an inductive definition. What is means is to look at each term $f_I \, dx_I$ of $\omega = \sum f_I \, dx_I$, compute the partial derivative of f_I in the usual way with respect to each variable and then attach the dx_I back on to each term of the partial derivative. Then move the differentials around if necessary.

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- ... The wedge product of two differential forms, written $\tau \wedge \omega$ or $\tau \cdot \omega$ ...

  When the definition for the wedge product of two forms is given, τ is assumed to be a p-form and ω is assumed to be a q-form. Thus $\tau \wedge \omega$ is a $(p+q)$-form.

- ... Note that $\tau \wedge \omega = (-1)^{\deg \tau \, \deg \omega} \omega \wedge \tau$ ...

  If τ is a p-form then we define $\deg \tau = p$.

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- $$\begin{aligned} \ldots d(\tau \cdot \omega) &= d(f_I g_J) \, dx_I \, dx_J = (df_I) g_J \, dx_I \, dx_J + f_I \, dg_J \, dx_I \, dx_J \\ &= (d\tau) \cdot \omega + (-1)^{\deg \tau} \tau \cdot d\omega \end{aligned}$$

  The last equality of the computation in Proposition 1.3 follows since dg_J will be a sum of partial derivatives with one new differential on each nonzero term so you need to move the new differential past $\deg \tau$ terms in $f_I dg_J dx_I dx_J$ to put it next to the dx_J terms to get $\tau \cdot d\omega$. Then factor out $(-1)^{\deg \tau}$ in each term.

- $$H^q_{DR}(\mathbb{R}^n) = \{\text{closed } q\text{-forms}\}/\{\text{exact } q\text{-forms}\}$$

  Precisely, we mean $H^q_{DR}(\mathbb{R}^n) = \ker d_q / \operatorname{im} d_{q+1}$.

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- $$H^q = \begin{cases} \mathbb{R} & q = 0 \\ 0 & q > 0 \end{cases}$$

  H^q means $H^q(\mathbb{R}^0)$ where $\mathbb{R}^0$ is the one point space. Notice that we have no local coordinates in this case.

- ...Since $(\ker \delta) \cap \Omega^0(\mathbb{R}^1)$ are the constant functions ...

  By $(\ker d) \cap \Omega^0(\mathbb{R}^1)$ we really mean $\ker d_0$ where the space we are working on is $\mathbb{R}^1$.

- $$H^0(U) = \mathbb{R}^m$$

  The reason why $H^0(U) = \mathbb{R}^m$ is because we can specify one constant function on each disjoint open interval.

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- ...Because $g(db) = d(gb) = dc = 0$ ...

  To see why $dc = 0$, recall that c is a closed form.

- ...diagram-chasing shows that this definition of d^* is independent of the choices made ...

  To see why d^* is independent of the choices made, notice that $f(da) = d(fa) = ddb = 0$ and since f is injective, a is well-defined.

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- $$H_c^1(\mathbb{R}^1) = \frac{\Omega_c^1(\mathbb{R}^1)}{\ker \int_{\mathbb{R}^1}} = \mathbb{R}^1$$

  The second equality follows by the first isomorphism theorem.

2 The Mayer-Vietoris Sequence

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- > ... The commutativity with d defines f^* uniquely:
  >
  > $$f^*(\sum g_I \, dy_{i1} \cdots dy_{i_q}) = \sum (g_I \circ f) \, df_{i_1} \cdots df_{i_q},$$
  >
  > where $f_i = y_i \circ f$ is the i-th component of the function f ...

  By df_{i_1} we mean in the sense of applying the exterior derivative to f_{i_1} and not in the sense of differentials. Thus,

  $$df_{i_1} = \sum_{j=1}^{m} \frac{\partial f_{i_1}}{\partial x_j} \, dx_j.$$

  So, $f^*(\sum g_I \, dy_{i1} \cdots dy_{i_q})$ is indeed a form in $\Omega^*(\mathbb{R}^m)$ with respect to the dx_is.

- We mention now here two properties of the pullback map which are used without mention throughout text and are very easy to prove. The first of which is the pullback distributes over wedge products. So, $f^*(\omega \wedge \tau) = f^*(\omega) \wedge f^*(\tau)$. The second is the pullback is anticommutative with respect to function composition in the sense that $(g \circ f)^*(\omega) = f^*(g^*(\omega))$ if $f : \mathbb{R}^m \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^k$ are smooth functions.

Page 20

- > ... Let $x_1, \ldots, x_n$ be the standard coordinate system and $u_1, \ldots, u_n$ a new coordinate system on $\mathbb{R}^n$, i.e., there is a diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $u_i = x_i \circ f = f^*(x_i)$ ...

  The diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ is defined by sending $x_i \mapsto u_i$ for all i and then extending linearly.

Page 21

- > ... The tangent space to M at p, written T_pM, is the vector space over $\mathbb{R}$ spanned by the operators $\partial/\partial x_1(p), \ldots, \partial/\partial x_n(p)$ ...

The $\partial/\partial x_i(p)$ are operators on smooth functions which sends a smooth function f to $(\partial/\partial x_i)(f(p))$.

- ...A differential form ω on M is a collection of forms ω_U for U ...

  By a form ω_U on U we mean a form on $\phi_\alpha(U_\alpha)$.

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- $$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^*(M) & \longrightarrow & \Omega^*(U) \oplus \Omega^*(V) & \longrightarrow & \Omega^*(U \cap V) & \longrightarrow & 0 \\ & & & & (\omega, \tau) & \mapsto & \tau - \omega & & \end{array}$$

  By $(\omega, \tau) \mapsto \tau - \omega$ we mean (ω, τ) is sent to the form $\tau - \omega$ restricted to $U \cap V$.

Page 25

- ...Since $-d(\rho_V \alpha)$ and $d\rho_U \alpha$ agree on $U \cap V$, they represent a global form on S^1 ...

  By a global form on S^1 we mean a form which is a generator. In this case we are looking for a generator of $H^1(S^1)$.

3 Orientation and Integration

Page 29

- On this page the phrase “top form” is used. By this we mean a form with the maximum amount of differentials attached.

- > ...We say that the altas is oriented if all the transition functions $g_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1}$ are orientation-preserving ...

  When we say the transition functions $g_{\alpha\beta}$ are orientation-preserving we mean in the sense defined on page 28. That is, the Jacobian of the determinant for $g_{\alpha\beta}$ is everywhere positive.

- > ...Let $\phi_\alpha : U_\alpha \xrightarrow{\sim} \mathbb{R}^n$ be a coordinate map. Then $\phi_\alpha^* \, dx_1 \cdots dx_n = f_\alpha \omega$ where f_α is a nowhere-vanishing real-valued function on U_α ...

  Firstly, $\phi_\alpha^* \, dx_1 \cdots dx_n$ really means the pullback of the top form $1 \, dx_1 \cdots dx_n$ by ϕ_α. Since ϕ_α is a diffeomorphism between $\mathbb{R}^n$ and U_α we can think of $\phi_\alpha^* \, dx_1 \cdots dx_n$ as just a change of coordinates of $1 \, dx_1 \cdots dx_n$. Now the space of top forms is one dimensional over the C^∞ functions as a module (the basis is $dx_1, \ldots, dx_n$). Hence at each point, the nowhere vanishing forms $\phi_\alpha^* \, dx_1 \cdots dx_n$ and ω differ by a nonzero constant $f_\alpha(x)$. This implies $\phi_\alpha^* \, dx_1 \cdots dx_n = f_\alpha \omega$ as forms.

- > ...any transition function $\phi_\beta \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ will pull $dx_1 \cdots dx_n$ to a positive multiple of itself ...

  To see this, use the definition of a pullback map and Exercise 3.1.

- > $$\sum_\alpha \int_{U_\alpha} \rho_\alpha \tau = \sum_{\alpha,\beta} \int_{U_\alpha} \rho_\alpha \chi_\beta \tau$$

To see this, notice

$$\sum_\alpha \int_{U_\alpha} \rho_\alpha \tau = \sum_\alpha \int_{U_\alpha} \sum_\beta \chi_\beta \rho_\alpha \tau = \sum_{\alpha,\beta} \int_{U_\alpha} \chi_\beta \rho_\alpha \tau = \sum_{\alpha,\beta} \int_{U_\alpha} \rho_\alpha \chi_\beta \tau.$$

We may interchange the integral and the sum by appealing to the analogous fact for Riemann integrals. To use this fact we need $\int_{U_\alpha} \sum_\beta |\chi_\beta \rho_\alpha \tau| < \infty$, but this is the case:

$$\int_{U_\alpha} \sum_\beta |\chi_\beta \rho_\alpha \tau| = \int_{U_\alpha} |\rho_\alpha \tau| < \infty,$$

since $\sum_\beta \chi_\beta = 1$ and $\rho_\alpha \tau$ has compact support in U_α since τ has compact support. We may exchange χ_β and ρ_α in the last equality because they are functions and not forms.

Page 31

- > ...since T maps the upper half plane to itself,
  >
  > $$\frac{\partial T_2}{\partial y_2}(y_1, 0) > 0$$

  We provide a little more detail. Since T maps the upper half plane to itself, $T_2(y_1, h) > 0$ for any $h > 0$. Hence

  $$\frac{\partial T_2}{\partial y_2}(y_1, 0) = \lim_{h \to 0} \frac{T_2(y_1, h) - T_2(y_1, 0)}{h} = \lim_{h \to 0} \frac{T_2(y_1, h)}{h} \geq 0.$$

  Strict inequality now follows because if we had equality then the determinant of the Jacobian of T would be equal to zero which is a contradiction to the assumption that the Jacobian is everywhere positive.

Page 33

- > ...where the last equality holds because the restriction of $g(x, y)\,dy$ to $\partial\mathbb{H}^2$ is 0 ...

  What we're really saying is $\int_{\partial\mathbb{H}^2} g(x, y)\,dy = 0$ by definition. More precisely, we're integrating over x but the form $g(x, y)\,dy$ has no differential dx attached so we define the integral to be zero. Generally, if we integrate a form over variables without the corresponding differentials attached, then the integral is defined to be zero.

4 Poincaré Lemmas

Page 34

- > ...it is enough to find a map K on $\Omega^*(\mathbb{R}^n \times \mathbb{R}^1)$ such that ...

  We should specify that K need be a linear map.

- > ...for $dK \pm Kd$ maps closed forms to exact forms ...

  We check that $dK \pm Kd$ takes closed forms to exact forms. If ω is a closed form then $d\omega = 0$, and

  $$(dK \pm Kd)(\omega) = dK\omega \pm Kd\omega = dK\omega.$$

  The form $dK\omega$ is exact because it's the image of $K\omega$ under d.

- > ...Every form on $\mathbb{R}^n \times \mathbb{R}$ is uniquely a linear combination of the following two types of forms:
  >
  > $$\text{(I)}(\pi^*\phi)f(x,t),$$
  > $$\text{(II)}(\pi^*\phi)f(x,t)\,dt,$$

  The reason this is true is because forms on $\mathbb{R}^n \times \mathbb{R}$ are real linear combinations of forms $f(x,t)g(x,t)\,dx_I$ and $f(x,t)g(x,t)\,dx_I\,dt$ where $I \subseteq \{1,\dots,n\}$ and $f(x,t)$ and $g(x,t)$ are forms on $\mathbb{R}^n \times \mathbb{R}$ (either could be the trivial form 1). Without loss of generality $g(x,t)dx_I$ either depends on $\mathbb{R}^n$ or on $\mathbb{R}^n \times \mathbb{R}$. In the former case we can we consider it as a form on $\mathbb{R}^n$ and call it ϕ (we drop the differentials here for brevity). Then $(\pi^*\phi) = g\,dx_I$ and we get types (I) and (II). In the latter case we abuse notation and write $f(x,t) = g(x,t)f(x,t)$ and take ϕ to be the identity form. This again gives forms of type (I) and (II). This idea can be directly generalized to compactly supported forms, compactly supported forms in the vertical direction, and for spaces $M \times N$ where M and N are arbitrary manifolds (this will be seen and used later in the text).

- $$-K d\omega = -K\left((d\pi^*\phi)f + (-1)^q \pi^*\phi\left(\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial t}\,dt\right)\right)$$

  Recall d and π^* commute if you don't see this equality quickly.

- $$d\omega = (\pi^* d\phi)f\,dt + (-1)^{q-1}(\pi^*\phi)\frac{\partial f}{\partial x}\,dx\,dt$$

  The reason we have $(-1)^{q-1}$ in the above equality is because we have to interchange the dt and dx to get $dx\,dt$.

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- **Corollary 4.1.2.1** Two manifolds with the same homotopy type have the same de Rham cohomology

  The corollary follows since the induced maps f^* and g^* are inverses.

Page 39

- ...the isomorphism $H_c^n(\mathbb{R}^n) \xrightarrow{\sim} \mathbb{R}$ is given by iterating π_* ...

  Indeed, notice π_* drops the degree by one each iteration. Also, notice that the inverse map is given by iterating e_* because it raises the degree by one each time.

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- ...a generator for $H_c^n(\mathbb{R}^n)$ is a bump n-form $\alpha(x)dx_1 \cdots dx_n$ ...

  The form $e_1(x_1)\cdots e_n(x_n)\,dx_1\cdots dx_n$ is an explicit generator.

5 The Mayer-Vietoris Argument

Page 42

- > . . . Let M be a manifold of dimension n . . .

  We are implicitly assuming M is a manifold without boundary in the definition of a good cover.

Page 43

- > . . . It is clear that J is also a direct set . . .

  Reflexively and transitivity are inherited, use contradiction to show J has the upper bound property

- > $$H^q(U \cup V) \simeq \ker r \oplus \operatorname{im} r \simeq \operatorname{im} d^* \oplus \operatorname{im} r$$

  If V is an infinite dimensional vector space and $f : V \to W$ is a linear map between vector spaces, then $V \simeq \ker f \oplus \operatorname{im} f$ is equivalent to AoC. Since we don't know *a priori* that $H^q(U \cup V)$ is finite dimensional, the first isomorphism is a consequence of AoC. Of course this doesn't cause any issues if we assume AoC.

Page 44

- > $$\int : H_q(M) \otimes H_c^{n-q}(M) \to \mathbb{R}$$

  This map is explicitly given by $\omega \otimes \tau \mapsto \int_{[M]} \omega \wedge \tau$.

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- $$\begin{array}{ccccccccc}
\cdots \longrightarrow & H^q(U \cup V) & \xrightarrow{\text{restriction}} & H^q(U) \oplus H^q(V) & \xrightarrow{\text{difference}} & H^q(U \cap V) & \xrightarrow{d^*} & H^{q+1}(U \cap V) & \longrightarrow \cdots \\
& \otimes & & \otimes & & \otimes & & \otimes & \\
\cdots \longleftarrow & H_c^{n-q}(U \cap V) & \xleftarrow{\text{sum}} & H_c^{n-q}(U) \oplus H_c^{n-q}(V) & \longleftarrow & H_c^{n-q}(U \cap V) & \xleftarrow{d^*} & H_c^{n-q-1}(U \cap V) & \\
& \Big\downarrow \int_{U \cup V} & & \Big\downarrow \int_U + \int_V & & \Big\downarrow \int_{U \cap V} & & \Big\downarrow \int_{U \cup V} & \\
& \mathbb{R} & & \mathbb{R} & & \mathbb{R} & & \mathbb{R} &
\end{array}$$

  Commutativity of the diagram means the integral of corresponding elements in either tensor product are equal when looking at a square of the diagram.

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- **Corollary 5.8** If M is a connected oriented manifold of dimension n, then $H_c^n(M) \simeq \mathbb{R}$. In particular if M is compact oriented and connected, $H^n(M) \simeq \mathbb{R}$

  To see why the corollary is true, recall that closed 0-forms on M are constant functions and then use Poincaré duality.

- $$\phi_\alpha : E|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times F$$

  By $E|_{U_\alpha}$ we really mean $\pi^{-1}(U_\alpha)$.

- ... and the transition functions are continuous functions with values in G:

  $$g_{\alpha\beta}(x) = \phi_\alpha \phi_\beta^{-1}|_{\{x\} \times F} \in G$$

  What we mean is that the transition function when restricted to $\{x\} \times F$ for any $x \in U_\alpha \cap U_\beta$ under the identification $\{x\} \times F \simeq F$ is a self-diffeomorphism of F and so looks like an element of G if G acts effectively on F (as explained in the remark on the same page). We associate this diffeomorphism with x and express this association as the map $g_{\alpha\beta} : B \to G$ given by $g_{\alpha\beta}(x) = \phi_\alpha \phi_\beta^{-1}|_{\{x\} \times F} \in G$.

- The trivial bundle is when $E = B \times F$, the π map is just projection onto the first factor, and the trivializations and transition functions are the identity. On another note, when E is a product space we call it a product bundle. To make things even more confusing, sometimes we refer to the trivial bundle as *the* product bundle. We mention these terms now, after the definition of a fiber bundle, since they are used without explicit definition in the remainder of the text.

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- > ...we will regard $M \times F$ as a product bundle over M ...

  Even though we regard $M \times F$ as a product bundle over M, this fact is not needed for the proof of the Künneth formula.

- > ...since tensoring with a vector space preserves exactness. Summing over all integers p yields the exact sequence ...

  It is a basic linear algebra fact that tensoring an exact sequence of vector spaces with another vector space preserves exactness (given that the maps act coordinate-wise). It's also a general fact that the direct sum of exact sequences of vector spaces is exact (again assuming the maps act coordinate-wise).

- > $$\psi d^*(\omega \otimes \phi) = \pi^*(d^*\omega) \wedge \rho^*\phi$$

  Note that d^* only acts on the first component of the tensor product by definition since the maps act coordinate-wise.

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- > ...and S a closed oriented submanifold of dimension k ...

  We are implicitly assuming S has no boundary.

Page 50

- > ...Since the pullback functions $\{\pi^*\rho_U, \pi^*\rho_V\}$ form a partition of unity on $(U \cup V) \times F$ subordinate to the cover $\{U \times F, V \times F\}$ ...

  More generally (which we prove below) if $\pi : E \to B$ is a fiber bundle and $\{\rho_\alpha\}$ is a partition of unity on B subordinate to the open cover $\{U_\alpha\}$, then $\{\pi^*\rho_\alpha\}$ is a partition of unity on E subordinate to $\{E_{U_\alpha}\}$. We tend to abuse notation and write ρ_α for $\pi^*\rho_\alpha$. Now for the proof. Clearly the support of $\pi^*\rho_\alpha$ is contained in $E|_{U_\alpha}$, so we need to show $\{\pi^*\rho_\alpha\}$ is a partition of unity. We first show $\sum_\alpha \pi^*\rho_\alpha = 1$ on E. Let $x \in E$ and $b = \pi(x)$. Then

  $$\sum_\alpha (\pi^*\rho_\alpha)(x) = \sum_\alpha (\rho_\alpha \circ \pi)(x) = \sum_\alpha \rho_\alpha(\pi(x)) = \sum_\alpha \rho_\alpha(b) = 1$$

  because $\sum_\alpha \rho_\alpha = 1$ on B. Now we cook up an open set about x on which $\sum_\alpha \pi^*\rho_\alpha$ is finite. Since $\{\rho_\alpha\}$ is a partition of unity on B, we can find an open set U about $\pi(x)$ where $\sum_\alpha \rho_\alpha$ is finite on U. We claim the desired set for x is $E|_U$. Indeed,

  $$\sum_\alpha (\pi^*\rho_\alpha)|_{E_U} = \sum_\alpha (\rho_\alpha \circ \pi)|_{E_U} = \sum_\alpha \rho_\alpha|_U$$

  which is finite by choice of U. This proves $\{\pi^*\rho_\alpha\}$ is a partition of unity on E subordinate to $E|_{U_\alpha}$.

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- > ...the Poincaré dual η_S is the unique cohomology class in $H^{n-k}(M)$ satisfying
  >
  > $$\int_S i^*\omega = \int_M \omega \wedge \eta_S$$

  η_s is unique because the image of $[\eta_s]$ under the isomorphism $(H_c^k(M))^* \simeq H^{n-k}(M)$ is by definition $\int_S i^*\omega$. However, this isomorphism is equivalent to the pairing $\int : H^q(M) \otimes H_c^{n-q}(M) \to \mathbb{R}$ being nondegenerate which is further equivalent to the map $[\tau] \mapsto \langle\, , \tau\rangle = \int_M \omega \wedge \tau$ being an isomorphism for $(H_c^k(M))^* \simeq H^{n-k}(M)$. Taking $[\tau] = [\eta_S]$, $[\eta_S]$ also maps to $\int_M \omega \wedge \eta_S$ and by injectivity these two integrals are equal.

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- > ...the compact Poincaré dual is the nontrivial class in $H^n_c(\mathbb{R}^n)$ represented by a bump form with totoal integral 1 ...

  To see this, recall that closed 0-forms are constants and then use Equation 5.14.

- > ...η_S is clearly the compact Poincaré dual of S in M because ...

  This is a typo. We mean to write η'_S.

6 The Thom Isomorphism

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- > ...whose fiber $\pi^{-1}(x)$ is a vector space ...

  We also required this vector space to be a fixed finite dimension n. This n is called the rank of the vector bundle.

- > $$\phi_\alpha : E|_{U_\alpha} = \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times \mathbb{R}^n$$

  These trivialization maps imply that $\dim(E) = n + \dim(M)$.

- > ...The maps
  >
  > $$\phi_\alpha \circ \phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \to (U_\alpha \cap U_\beta) \times \mathbb{R}^n$$
  >
  > are vector-space isomorphisms of $\mathbb{R}^n$ in each fiber and hence give rise to maps
  >
  > $$g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})$$
  >
  > $$g_{\alpha\beta}(x) = \phi_\alpha \phi_{\beta^{-1}}|_{\{x\} \times \mathbb{R}^n}$$

  In particular, this means the first coordinate map $(U_\alpha \cap U_\beta) \to (U_\alpha \cap U_\beta)$ is the identity map and the second coordinate map is given by $g_{\alpha\beta}$. That is, (x, y) is mapped to $(x, g_{\alpha\beta}(x)y)$.

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- > ...every vector bundle has a well-defined global zero section ...

  The idea behind why this is true is we can choose a canonical zero in each fiber $\pi^{-1}(x)$ and this choice of zero is continuous because the transition maps are linear fiber-wise. Precisely, s is locally defined by $s(x) = \phi_\alpha^{-1}(x, 0)$. These local maps are continuous since ϕ_α is and glue together to give a global map on M because $\phi_a \circ \phi_\beta^{-1}$ is a linear automorphism on each fiber.

- > ... A collection of sections $s_1, \dots, s_n$ over an open set U in M is a frame on U if for every point x in U, $s_1(x), \dots, s_n(x)$ form a basis of the vector space $E_x = \pi^{-1}(x)$ ...

  There always exists a smooth local frame on a vector bundle, and in fact this is equivalent to a local trivialization. Indeed, let $e_1, \dots, e_n$ be the standard basis vectors on $\mathbb{R}^n$. Let $\phi_\alpha : E_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{R}^n$ be the trivialization for U_α, and define $s_i : U \to E$ by $s_i(x) = \phi_\alpha^{-1}(x, e_i)$. We claim $\{s_i\}_{1 \leq i \leq n}$ is a smooth frame on U_α. Since $\phi_\alpha^{-1}(x, e_i) \in E_x$, $\pi \circ s_i$ is the identity on U_α. s_i is also smooth because ϕ_α is a diffeomorphism, so in particular smooth with smooth inverse. The standard basis for $\{x\} \times \mathbb{R}^n$ is obviously $(x, e_i), \dots, (x, e_n)$, and since $\phi_\alpha|_{E_x} : E_x \simeq \{x\} \times \mathbb{R}^n$ is a linear isomorphism, $\phi_\alpha^{-1}(x, e_i), \dots, \phi_\alpha^{-1}(x, e_k)$ is a basis for E_x. Since i and x were arbitrary, this proves $\{s_i\}_{1 \leq i \leq n}$ is a smooth frame on U_α. The U_α cover M so we conclude there exists a local smooth frame on the bundle. Conversely, suppose there exists a local smooth frame on U_α. Let $\{s_i\}$ be the smooth frame on U_α. Then for every $x \in U_\alpha$ and every $v_x \in E_x$ we may uniquely write $v_x = \sum_{i=1}^n \alpha_i s_i(x)$ for some $\alpha_i \in \mathbb{R}$. We claim the map

  $$\phi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^n \quad \phi_\alpha(v_x) = (x, \alpha_1, \dots, \alpha_n),$$

  where $v_x \in E_x \subset E|_{U_\alpha}$, is a trivialization. It is clearly bijective since the expression $v_x = \sum_{i=1}^n \alpha_i s_i(x)$ is unique. It is also bi-continuous because the component maps can be realized as projections in either direction. Since projections are smooth this means ϕ_α is also smooth and hence a trivialization for U_α. The U_α cover M so we get a trivialization for E. The importance of this converse is that we may have a frame which does not correspond to the original trivializations for the bundle and so we can find a different trivialization for the bundle.

- > $$g_{\alpha\beta} = \lambda_\alpha g'_{\alpha\beta} \lambda_\beta^{-1} \quad \text{on} \quad U_\alpha \cap U_\beta$$

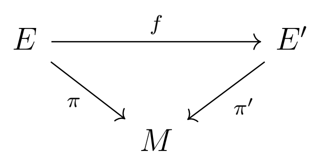
  By $\lambda_\alpha g'_{\alpha\beta} \lambda_\beta^{-1}$ we mean multiplication in the group $GL(n, \mathbb{R})$ and not function composition. Also, by λ_β^{-1} we mean apply λ_β and then take the inverse element in $GL(n, \mathbb{R})$. In other words, $\lambda_\beta^{-1}(x) = (\lambda_\beta(x))^{-1}$.

- > ... two trivializations differ by a nonsingular transformation of $\mathbb{R}^n$ at each point ...

  This is because each trivialization is a linear isomorphism on each fiber.

- > ... a bundle map, is a fiber-preserving smooth map $f : E \to E'$ which is linear on corresponding fibers ...

  An equivalent definition of a bundle map is a smooth map $f : E \to E'$ which makes the following diagram commutative:



Page 55

- > ... The diffeomorphism
  > $$\psi_\alpha : U_\alpha \xrightarrow{\sim} \mathbb{R}^n$$
  > induces a map
  > $$(\psi_\alpha)_* : T_{U_\alpha} \xrightarrow{\sim} T_{\mathbb{R}^n}$$

  The induced map is precisely precomposition with ψ_α^{-1}.

Page 56

- > ... Functorial operations on vector spaces carry over to vector bundles. For instance, if E and E' are vector bundles over M of rank n and m respectively, their direct sum $E \oplus E'$ is a vector bundle over M whose fiber at the point x in M is $E_x \oplus E'_x$ ... Similarly we can define the tensor product $E \otimes E'$, the dual E^*, and $\mathrm{Hom}(E, E')$ ...

All of these spaces as sets can be defined to be the disjoint union of the fibers with the corresponding operator applied to the fiber. So

$$E \oplus E' = \bigsqcup_{x \in M} E_x \oplus E'_x, \qquad E^* = \bigsqcup_{x \in M} E^*_x,$$
$$E \otimes E' = \bigsqcup_{x \in M} E_x \otimes E'_x, \qquad E^* \otimes E' = \bigsqcup_{x \in M} E^*_x \otimes E'_x,$$

where we identify $\mathrm{Hom}(E, E')$ with $E^* \otimes E'$. The corresponding π maps send a point to its index so the fibers are exactly the disjoint sets as desired. The topology on these spaces is defined to be the finest topology such that the trivializations are continuous where the trivialization maps are build from those of E and E' with the associated operation. The smooth manifold structure is given by letting the trivializations be those in the smooth atlas and its a quick check to show the transition functions for the atlas are smooth.

- > ... This bundle $f^{-1}E$ is defined to be the subset of $N \times E$ given by
  >
  > $$\{(n, e) \mid f(n) = \pi(e)\}$$

  The topology on $f^{-1}E$ is the subspace topology of $N \times E$. The associated π map is projection onto the first factor. The trivialization maps are as follows. If ϕ_α is a trivialization on $E|_{U_\alpha}$, then ψ_α is a trivialization on $f^{-1}(U_\alpha)$ given by $\psi_\alpha(n, e) = (n, \mathrm{proj}_2(\phi_\alpha(e)))$ where $\mathrm{proj}_2(\phi_\alpha(e))$ is projection onto the second factor of $\phi_\alpha(e)$. This implies the rank of $f^{-1}E$ is the same as the rank of E. The smooth manifold structure is given by letting the trivializations be those in the smooth atlas and its a quick check to show that the transition functions for the atlas are smooth.

- At this point we would like to mention two other methods of creating new vector bundles from old ones. Namely subbundles and quotient bundles. We speak of subbundles first. If $\pi : E \to M$ is a vector bundle, then a subbundle $E' \subseteq E$ over M is a collection of liner subspaces $E'_x \subseteq E_x$ that make up a vector bundle over M in their own right. Given a subbundle E' of rank k' of the vector bundle E of rank k we may form the quotient bundle E/E'. We start by defining an equivalence relation on E such that $x \sim y$ if and only if x and y are in the same fiber with respect to E and $x - y$ is in the corresponding fiber with respect to E'. From now on we write E/E' for $E/\sim$. This equivalence relation induces a bundle projection $\pi : E/E' \to M$. It's a quick check to verify the fiber over x is the vector spaces E_x/E'_x. Since E' is a subbundle, it has a local trivialization, which is equivalent to a local smooth frame. Now notice that if ϕ_α and ϕ'_α are the trivializations for U_α on E and E' respectively, then $\phi'_\alpha = \phi_\alpha|_{E'_{U_\alpha}}$ by definition. So, the sections $s_i(x) = \phi_\alpha^{-1}(x, e_i)$ for $1 \leq i \leq k$ are sections on U_α which form a smooth frame on U_α for E that extends the frame on U_α for E'. This means we can extend a smooth local frame on E' to a smooth local frame on E. The extended frame gives rise to trivializations for E which extend those for E' by construction of the correspondence between local smooth frames and trivializations. Call these extended trivializations ϕ''_α. Then ϕ''_α induces a trivialization $E/E'|_{U_\alpha} \simeq U_\alpha \times \mathbb{R}^{k-k'}$ by ignoring the first k coordinates of $\mathbb{R}^{k-k'}$. The smooth structure on E/E' is given by letting these trivializations be those in the smooth atlas and it's a quick check to show the corresponding transition functions for the atlas are smooth.

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- > ...Since a product bundle pulls back to a product bundle we see that $f^{-1}(E)$ is locally trivial, and is therefore a vector bundle ...

  This is quite a circular and poor explanation of why the pullback bundle is indeed a vector bundle. Refer to the notes for page 56 about the construction of the pullback vector bundle and the following note for a more concrete explanation.

- > ...The fiber $f^{-1}E$ over a point y in N is isomorphic to $E_{f(y)}$ ...

  Let $\text{proj}_1 : f^{-1}E \to N$ be projection onto the first factor. Then this statement follows because $\text{proj}_1^{-1}(y) = \{y\} \times E_{f(y)}$ by commutativity of the previous diagram in the text.

- > ...if we have a composition
  >
  > $$M'' \xrightarrow{g} M' \xrightarrow{f} M,$$
  >
  > then $(g \circ f)^{-1}E = g^{-1}(f^{-1}M)$

While these spaces as sets are literally the same, this equality requires some checks such as verifying that the topology on the spaces is the same, the trivialization maps are identical as well as the smooth atlases. However all these checks are fairly straightforward and follow by composing functions.

- > ... $\text{Vect}_k()$ becomes a functor from the category of manifolds and smooth maps to the category of pointed sets and base point preserving maps.

  The pullback map f^{-1} being base point preserving means exactly that the pullback of the trivial bundle is trivial. Indeed, if $\pi : M \times \mathbb{R}^k \to M$ is the trivial bundle and $f : N \to M$ is continuous, then $f^{-1}E$ is the set of tuples (n, m, x) in $N \times M \times \mathbb{R}^k$ such that $f(n) = m$ which means M is redundant, so our total space is $N \times \mathbb{R}^k$. The bundle projection $\pi : N \times \mathbb{R}^k \to N$ is projection onto the first factor, so $f^{-1}(M \times \mathbb{R}^k)$ is the trivial rank k bundle over N.

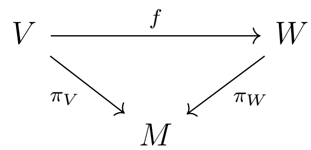
- > ... $\text{Hom}(V, W) = V^* \otimes W$ is a vector bundle over B whose fiber at p consists of all the linear maps from V_p to W_p

  To see this notice that the fiber in $V^* \otimes W$ over p is exactly $V_p^* \otimes W_p$. Then look at the image of $V_p^* \otimes W_p$ under the isomorphism $V^* \otimes W \xrightarrow{\sim} \text{Hom}(V, W)$ which sends a simple tensor $f \otimes w$ to the map $fw : V \to W$ defined by $fw(v) = f(v)w$.

Page 58

- > ... $\text{Iso}(V, W)$ inherits a topology from $\text{Hom}(V, W)$, and is a fiber bundle with fiber $GL(n, \mathbb{R})$. An isomorphism between V and W is simply a section of $\text{Iso}(V, W)$ ...

  This is an example of a vector subbundle (a collection of linear subspaces of the fibers of the original bundle that form a vector bundle in their own right). It is also not outright obvious that $\text{Iso}(V, W)$ forms a vector bundle over B, but all of the necessary checks are fairly straightforward.
  We now prove that an isomorphism between V and W is exactly a global section of $\text{Iso}(V, W)$. Suppose first we have a section $s : B \to \text{Iso}(V, W)$. We want to cook up a continuous map $f : V \to W$ such that the following diagram is commutative:



Define f locally by $f|_{V_p} = s(p) : V_p \xrightarrow{\sim} W_p$. Since the fibers V_p are mutually disjoint, f is well-defined. It is also continuous since each $s(p)$ is continuous (this can also be checked directly from the definition of continuity). Since each $s(p)$ is an isomorphism, so is f. Indeed, if $v \in V$ and lies in the fiber over p, then $f = s(p)$ on v (really V_p) and so $\pi_V(v) = \pi_W(s(p)(v)) = \pi_W(f(v))$ because $s(p)$ is a bundle isomorphism. Conversely, if $f : V \xrightarrow{\sim} W$ is an isomorphism, then define $s : B \to \mathrm{Iso}(V, W)$ by $s(p) = f|_{V_p} : V \to W$. $s(p)$ is a bundle map onto its image and its image is exactly W_p by commutative of the diagram above.

- > ...Since $f_{t_0}^{-1}E \simeq F$, $\mathrm{Iso}(f^{-1}E, \pi^{-1}F)$ has a section over $Y \times t_0$, which a priori is also a section of $\mathrm{Hom}(f^{-1}E, \pi^{-1}F)$ ...

  The fact that an isomorphism between V and W is simply a global section of $\mathrm{Iso}(V, W)$ extends more generally to the fact that a section of $\mathrm{Iso}(V, W)$ over $U \subseteq B$ is exactly an isomorphism between the restricted bundles $V|_U$ and W_U. This section is also a section of $\mathrm{Hom}(V, W)$ because $\mathrm{Iso}(V, W)$ is a subbundle (this can also be checked directly from the definition).

- > ...any linear map near an isomorphism remains an isomorphism ...

  This statement becomes clear in the following reformulation. These isomorphisms are linear maps and a linear map is an isomorphism if and only if the determinant of the corresponding matrix is nonzero. Now the determinant is a continuous map on the space of matrices so small enough open sets about these isomorphisms contain maps which have nonzero determinant and so are themselves isomorphisms.

Page 59

- > ...Since $g^{-1}E$ is a vector bundle on a point, it is trivial ...

This is because we have one trivialization map $\phi : M \xrightarrow{\sim} \{p\} \times \mathbb{R}^n$ for some n which precisely says the bundle is trivial.

Page 60

- > ...The typical transition function of this atlas,
  >
  > $$(\psi_\alpha \times 1) \circ \phi_\alpha \phi_\beta^{-1} \circ (\psi_\beta^{-1} \times 1) : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \times R^n$$
  >
  > sends (x, y) to $(h_{\alpha\beta}(x), g_{\alpha\beta}(\psi_\alpha^{-1}(x))y)$ and has Jacobian matrix
  >
  > $$\begin{pmatrix} D(h_{\alpha\beta}) & * \\ 0 & g_{\alpha\beta}(\psi_\alpha^{-1}(x)) \end{pmatrix}$$
  >
  > where $D(h_{\alpha\beta})$ is the Jacobian matrix of $h_{\alpha\beta}$ ...

We give a little more detail to verify this matrix is actually the Jacobian of the map $(\psi_\alpha \times 1) \circ \phi_\alpha \phi_\beta^{-1} \circ (\psi_\beta^{-1} \times 1)$. First notice x and y are vectors in m and n variables respectively. Similarly $h_{\alpha\beta}(x)$ and $g_{\alpha\beta}(\psi_\alpha^{-1}(x))y$ are functions in m and n variables respectively. We will pick the ordered basis $x_1, \ldots, x_m, y_1, \ldots, y_n$ for our matrix. Then we may write our matrix in blocks as

$$\begin{pmatrix} D_{m,m} & D_{m,n} \\ D_{n,m} & D_{n,n} \end{pmatrix},$$

where the subscripts on each block indicates its size. By the definition of the Jacobian, the entries of $D_{m,m}$ are the partial derivatives of the first m component functions of the map $(x, y) \mapsto (h_{\alpha\beta}(x), g_{\alpha\beta}(\psi_\alpha^{-1}(x))y)$ (which are the components of $h_{\alpha\beta}$) with respect to the x_i's (by the choice of ordered basis). This is exactly $D(h_{\alpha\beta})$, so $D_{m,m} = D(h_{\alpha\beta})$. Similarly, the entries of $D_{n,m}$ are just the partial derivatives of the component functions of $h_{\alpha\beta}(x)$ with respect to the y_i's and are therefore 0 (because $h_{\alpha\beta}(x)$ is not a function of the y_i's). Hence $D_{n,m} = 0$. This makes the block matrix upper triangular so we don't need to calculate $D_{m,n}$ to find the determinant and therefore set $D_{m,n} = *$. Lastly, the entries of $D_{n,n}$ are the partial derivatives of the component functions of $g_{\alpha\beta}(\psi_\alpha^{-1}(x))y$ with respect to the y_i's. Now $g_{\alpha\beta}(\psi_\alpha^{-1}(x))$ is an $n \times n$ matrix, call it A. By comparing the definitions of the Jacobian and matrix differentiation and recalling the fact $\partial Ay/\partial y = A$ (as matrices) we conclude $D_{n,n} = A = g_{\alpha\beta}(\psi_\alpha^{-1}(x))$ as desired. This proves the matrix is of the form described.

- **Proposition 6.13.** If $\pi : E \to M$ is an orientable vector bundle and M is orientable of finite type, then $H_c^*(E) \simeq H_c^{*-n}(M)$.

This follows from Lemma 6.12 and the argument at the top of page 60 once we prove that if M has a finite good cover then so does E. Letting $\{U_\alpha\}$ be the open cover on M for the bundle, and $\{V_i\}_{1\le i\le n}$ a finite good cover for M, we may take $\{U_\alpha\}$ to be $\{V_i\}_{1\le i\le n}$. Indeed, since $\{V_i\}_{1\le i\le n}$ is a good cover, the V_is are diffeomorphic to $\mathbb{R}^m$, hence contractible, implying $E|_{V_i}$ is the trivial bundle. The corresponding trivializations are linear isomorphisms on fibers since on fibers these maps are the same as the trivializations for the U_αs on fibers. We claim $\{E|_{V_i}|\}_{1\le i\le n}$ is a finite good cover for E. Since π is a continuous surjection, $\{E|_{V_i}\}_{1\le i\le n}$ is a finite cover for E. It is a good cover since

$$\bigcap_{j=1}^{p} E|_{V_{i_j}} = E|_{\bigcap_{j=1}^{p} V_{i_j}} \simeq E|_{\mathbb{R}^m} \simeq \mathbb{R}^m \times \mathbb{R}^k \simeq \mathbb{R}^{m+k},$$

where $E|_{\mathbb{R}^m} \simeq \mathbb{R}^m \times \mathbb{R}^k$ because $\bigcap_{j=1}^{p} V_{i_j} \simeq \mathbb{R}^m$ is contained in some V_i (actually any of the V_{i_j}).

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- ...although a form in $\Omega_{cv}^*(E)$ need not have compact support in E, its restriction to each fiber has compact support ...

These definitions are actually equivalent. Since the forward implication is proved, we prove the reverse implication. Since E is a smooth manifold, there exists a Riemannian metric $\langle\ ,\ \rangle$ on E. By assumption $E|_x \cap \operatorname{Supp}\omega$ is compact, hence bounded if we endow E with the metric. Now define the support distance function $d : M \to \mathbb{R}$ by $d(x) = \sup\{\sqrt{\langle y, y\rangle} \mid y \in E_x \cap \operatorname{Supp}\omega\}$. So, d sends x to the largest distance from elements in its fiber to the origin (recall there is a natural choice of origin here because $E|_x$ is a vector space). Also, d is continuous since ω is continuous. Now let K be compact in M and cover K by finitely many open balls B_i for $1 \le i \le n$. Then $E|_{B_i} \simeq B_i \times \mathbb{R}^k \subset \mathbb{R}^{m+k}$. Since the B_is are compact and convex, $d|_{B_i}$ is bounded, hence $E|_{B_i} \cap \operatorname{Supp}\omega$ is a bounded and closed set (recall the B_is are compact hence closed) and therefore by Heine-Borel compact. $E|_K \cap \operatorname{Supp}\omega = \bigcup_{i=1}^{n} E|_{B_i} \cap \operatorname{Supp}\omega$ is a finite union of compact sets, hence compact.

- > $\ldots t_1, \ldots, t_n$ and $u_1, \ldots, u_n$ be the fiber coordinates on $E|_{U_\alpha}$ and $E|_{U_\beta}$ given by ϕ_α and ϕ_β respectively ...

  By fiber coordinates we just mean $t_1, \ldots, t_n$ and $u_1, \ldots, u_n$ are the coordinate functions for ϕ_α and ϕ_β respectively.

- > $\ldots$ a form ω in $\Omega^*_{cv}(E)$ is locally of type (I) or type (II) ...

  It's important to note that if a linear term of ω locally looks like a form of type (I) or (II), then it locally looks like type (I) or (II) everywhere. This is because when we view that term locally we apply a change of coordinates to view it as a form on $U_\alpha \times \mathbb{R}^n$ and this obviously doesn't add or take off any differentials.

Page 62

- > $\ldots \{\rho_\alpha\}$ a partition of unity subordinate to $\{U_\alpha\}$, and ω a form in $\Omega^*_{cv}(E)$. Since $\omega = \sum \rho_\alpha \omega$ ...

  When we write $\omega = \sum \rho_\alpha \omega$ we really mean $\omega = \sum \pi^* \rho_\alpha \omega$ where $\{\pi^* \rho_\alpha\}$ is the partition of unity on E subordinate to the cover $\{E|_{U_\alpha}\}$.

- > $\ldots$ we may assume E to be the product bundle $M \times \mathbb{R}^n$ ...

  Since ρ_α has its support in $E|_{U_\alpha}$ we may consider $\rho_\alpha \omega$ as a form on $E|_{U_\alpha} \simeq U_\alpha \times \mathbb{R}^n$. Thus we may assume $\rho_\alpha \omega$ is a form on the trivial bundle, and we abuse notation by writing $M = U_\alpha$ so we may assume E to be the product bundle $M \times \mathbb{R}^n$.

Page 63

- > $$\int_M \tau \wedge \pi_* \omega = \sum_\alpha \int_{U_\alpha} \tau \wedge \pi_*(\rho_\alpha \omega)$$

This requires a quick check. By the definition of integration of forms over manifolds we have

$$\int_M \tau \wedge \pi_* \omega = \sum_\alpha \int_{U_\alpha} \tau \wedge \rho_\alpha \pi_* \omega.$$

So what we need to show is $\pi_*(\rho_\alpha \omega) = \rho_\alpha \pi_*(\omega)$ locally (that is on U_α), and thus we may assume we are working over the trivial bundle. The equality obviously holds for forms of type (I) so assume ω is of type (II). In this case,

$$\rho_\alpha \pi_*(\omega) = \rho_\alpha \phi \int_{\mathbb{R}^n} f(x,t)\, dt_1 \dots dt_n$$

By unabusing the notation and no longer writing $\rho_\alpha \circ \pi$ for ρ_α, we see $\rho_\alpha \circ \pi = \pi^* \rho_\alpha$ by definition of π^*. Hence

$$\pi_*(\rho_\alpha \omega) = \pi_*((\rho_\alpha \circ \pi)(pi^* \phi) f(x,t)\, dt_1 \dots dt_n) = \rho_\alpha \phi \int_{\mathbb{R}^n} f(x,t)\, dt_1 \dots dt_n.$$

This proves $\pi_*(\rho_\alpha \omega) = \rho_\alpha \pi_*(\omega)$ for type (II) forms, and the integral identity follows.

- **Proposition 6.16** (Poincaré Lemma for Compact Vertical Supports). Integration along the fiber defines an isomorphism

  $$\pi_* : H^*_{cv}(M \times \mathbb{R}^n) \to H^{*-n}(M).$$

  We must also assume M is of finite type.

Page 64

- $$\pi_* d^* \omega = \pi_*((\pi^* d\rho_U) \cdot \omega) = (d\rho_U) \cdot \pi_* \omega = d^* \pi_* \omega$$

  Let's give a little more detail to this computation. First notice that we assume ω is a closed form. The first equality is slightly confusing because we drop the abusive notation of writing $\{\rho_\alpha\}$ to mean a partition of unity on M and to mean the pullback partition of unity on E by $\pi : E \to M$. If we drop this abusive notation then

  $$\pi_* d^* \omega = \pi_*(d(\pi^* \rho_U \omega)) = \pi_*((\pi^* d\rho_U) \cdot \omega + (\pi^* \rho_U) \cdot d\omega) = \pi_*((\pi^* d\rho_U) \cdot \omega),$$

because d and π^* commute, $\deg \rho_U = 0$ as it is a function, and ω is closed. This verifies the first equality, the second is by the projection formula, and the third is proved by computing $d^* \pi_* \omega$ in similar spirit to the computation above.

- > ...if U is diffeomorphic to $\mathbb{R}^n$, then $E|_U$ is trivial, so that in this case the Thom isomorphism reduces to the Poincaré lemma for compact vertical supports ...

  We mean the Thom isomorphism reduces to the Poincaré lemma for compact vertical supports on $E|_U$.

- > ...the Thom isomorphism, which is inverse to π_*, is given by
  >
  > $$\mathscr{T}(\quad) = \pi^*(\quad) \wedge \Phi$$

  As another remark, we don't need to check $\pi^*(\pi_* \omega) \wedge \Phi = \omega$ because we know $\mathscr{T}$ is an isomorphism, hence has a unique left and right inverse which are equal, and π_* is a left inverse since $\pi_*(\pi^* \omega \wedge \Phi) = \omega$.

- > PROOF. Since $\pi_* \Phi = 1$, $\Phi|_{\text{fiber}}$ is a bump form on the fiber with total integral 1. Conversely, if Φ' in $H^n_{cv}(E)$ restricts to a generator on each fiber, then
  >
  > $$\pi_*((\pi^* \omega) \wedge \Phi') = \omega \wedge \pi_* \Phi' = \omega.$$
  >
  > Hence $\pi^*(\quad) \wedge \Phi'$ must be the Thom isomorphism $\mathscr{T}$ and $\Phi' = \mathscr{T}(1)$ is the Thom class. □

  Let's give a little more detail to this proof. If $\pi_* \Phi = 1$, then Φ cannot locally be a type (I) form so it must be type (II). So, if we restrict Φ to a fiber we have

  $$\Phi = (\pi^* \phi) f(x,t)\, dt_1 \dots dt_n.$$

  It's by definition smooth and has compact support since $f(x,t)$ does on fibers. Hence it's a bump form and therefore a generator of $H^n_c(F)$. The total integral is 1 since $\pi_* \Phi = 1$. Now if Φ' restricts to a generator on each fiber, then on fibers we can consider it as a bump form with total integral 1. This means

$\pi_*\Phi' = 1$ on each fiber and since the fibers cover E we conclude $\pi_*\Phi' = 1$ on E. With the projection formula, this shows

$$\pi_*((\pi^*\omega) \wedge \Phi') = \omega \wedge \pi_*\Phi' = \omega.$$

So, π_* is a left inverse for $\pi^*(\quad) \wedge \Phi'$ which by uniqueness implies $\pi^*(\quad) \wedge \Phi'$ is the Thom isomorphism. Then $\mathscr{T}(1) = \pi^*(1) \wedge \Phi' = \Phi'$ is the Thom class.

Page 65

- > ...the Thom class of $E \oplus F$ is ...

  It's a general fact (and straightforward to show) that the direct sum of two orientable vector bundles is orientable. So, it makes sense to talk about the Thom class of $E \oplus F$.

- > ... $\pi_1^*\Phi(E) \wedge \pi_2^*\Phi(F)$ is a class in $H_{cv}^{m+n}(E \oplus V)$ ...

  The projection functions are smooth which ensures $\pi_1^*\Phi(E) \wedge \pi_2^*\Phi(F)$ is actually a form. It has compact support on fibers because $\Phi(E)$ and $\Phi(F)$ do. Indeed, let $x \in M$, $(E \oplus F)_x = E_x \oplus F_x$, U_E be the compact support for $\Phi(E)$ in E_x, and similarly for $\Phi(F)$. Then $U_E \oplus U_F \subset E_x \oplus F_x$ is the compact support of $\pi_1^*\Phi(E) \wedge \pi_2^*\Phi(F)$ on $(E \oplus F)_x$.

- > ... $\pi_1^*\Phi(E) \wedge \pi_2^*\Phi(F)$ is a class in $H_{cv}^{m+n}(E \oplus F)$ whose restriction to each fiber is a generator of the compact cohomology of the fiber, since the isomorphism
  >
  > $$H_c^{m+n}(\mathbb{R}^m \times \mathbb{R}^n) \simeq H_c^m(\mathbb{R}^m) \otimes H_c^n(\mathbb{R})$$
  >
  > is given by the wedge product of the generators ...

  We provide a little more detail. If we restrict $\pi_1^*\Phi(E) \wedge \pi_2^*\Phi(F)$ to the fiber $(E \oplus F)_x = E_x \oplus F_x$ for $x \in M$, we may consider $\pi_1^*\Phi(E) \wedge \pi_2^*\Phi(F)$ as a form on $\mathbb{R}^m \times \mathbb{R}^n$ because $E_x \oplus F_x \simeq \mathbb{R}^m \times \mathbb{R}^n$ (recall $\oplus$ is the same as $\times$ for finite products and sums). Since we are restricting our form to a fiber, compact support in the vertical direction is simply compact support. So, we can view

$\pi_1^*\Phi(E) \wedge \pi_2^*\Phi(F)$ as a form on $H_c^{m+n}(\mathbb{R}^m \times \mathbb{R}^n)$. It's straightforward to show there is an isomorphism

$$H_c^{m+n}(\mathbb{R}^m \times \mathbb{R}^n) \simeq H_c^m(\mathbb{R}^m) \otimes H_c^n(\mathbb{R})$$

given by taking the direct sum of pullbacks, and the wedge sum respectively. Since isomorphisms take generators to generators, $\pi_1^*\Phi(E)\wedge\pi_2^*\Phi(F)$ is a generator since it's mapped to $\Phi(E)\otimes\Phi(F)$ which is a generator of $H_c^m(\mathbb{R}^m)\otimes H_c^n(\mathbb{R}^n)$ because $\Phi(E)$ and $\Phi(F)$ are themselves generators.

Page 66

- > ...If S is a submanifold in M, the normal bundle $N = N_{S/M}$ of S in M is the vector bundle on S defined by the exact sequence
  >
  > $$0 \to T_S \to T_M|_S \to N \to 0,$$
  >
  > where $T_M|_S$ is the restriction of the tangent bundle of M to S.

  First note, by "the restriction of the tangent bundle of M to S" we mean the pullback bundle i^*T_M where $i : S \to M$ is the immersion of S in M (an immersion is a differentiable function between differentiable manifolds whose derivative is everywhere injective for example an embedding) defining S as a submanifold of M. Now this is quite a poor way to define the normal bundle, because it does not shed any light on how the bundle is constructed and even if it exists. We give a more constructive discussion here. Let then the normal bundle $N_{S/M}$ of S in M is defined as $N = T_M|_S/T_S$. T_S can be realized as a subbundle of T_M by the inclusion map, so this quotient makes sense. It's then a quick check to show the exactness of

  $$0 \to T_S \to T_M|_S \to N \to 0.$$