k-Schur functions

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Outline

Back of Pocket

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k-Schur functions $s_\lambda^{(k)}$ are sums of Schur functions over a special set of tableaux $\mathbb{A}_\lambda^{(k)}$ with coefficients in $\mathbb{Z}[t]$.

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Conventions

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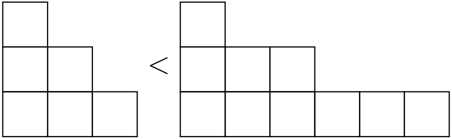

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then $w = 544122$.

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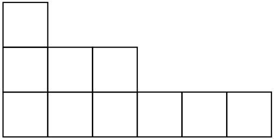

is 6-bounded and not 5-bounded.

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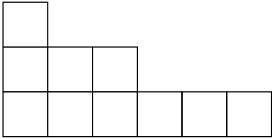

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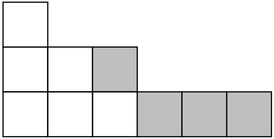

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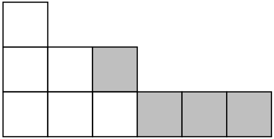

Horizontal (vertical) strips **do not** need to lie in a single row (column).

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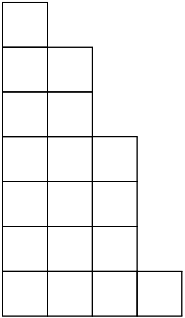

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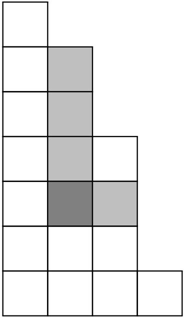

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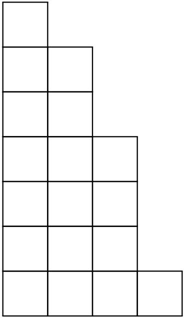

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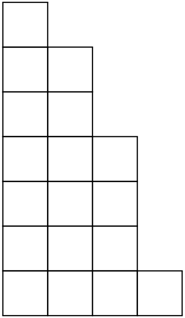

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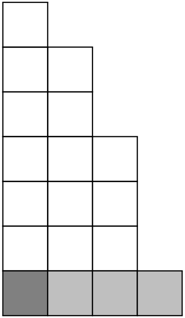

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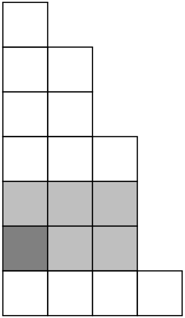

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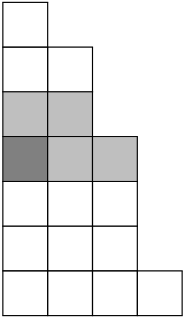

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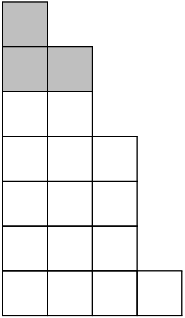

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- Construct an index vector I according to the rule:

$$I_1 = 0 \quad \text{and} \quad I_r = \begin{cases} I_{r-1} + 1 & \text{if } r \text{ is east of } r-1, \\ I_{r-1} & \text{otherwise.} \end{cases}$$

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- Remove the labels in the cells corresponding to this sequence and repeat.
- $\mathrm{charge}(T)$ is the sum of the entries of all I.

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4	5				
3	4				
2	2	3	5		
1	1	1	2	3	7

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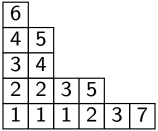

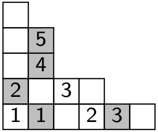

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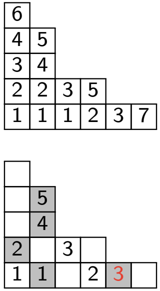

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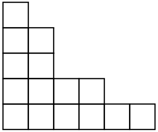

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$$I = [0, 0, 0, 0, 1, 1, 2], \quad I = [0, 0, 1, 1, 1], \quad \text{and} \quad I = [0, 1, 1].$$

So $\mathrm{charge}(T) = 9$.

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- Place the new word back into $\mathrm{shape}(T)$.

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Consider

4	5	5						
3	3	4	6	6				
2	2	3	3	3	5	6		
1	1	1	1	2	2	3	3	4

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$$\begin{matrix} 4334333334 \\ ())()))))(\end{matrix} \rightarrow \begin{matrix} 4334344444 \\ ())()(((((\end{matrix}$$

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$$4334333334 \to 4334344444$$

( )) ( ))))) (( )) ( ) (((((

Producing

4	5	5						
3	3	4	6	6				
2	2	3	4	4	5	6		
1	1	1	1	2	2	4	4	4

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$\mathbb{B}_3$ acts on

3	
2	
1	1

by returning the following set tableaux:

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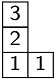
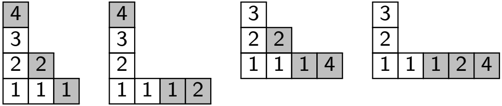
$\mathbb{B}_3$ acts on  by returning the following set tableaux:



The Schensted Algorithm

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- If k is larger than the largest entry in T_1, then $k \to T$ is given by appending k at the end of T_1.
- Otherwise, find the leftmost entry in T_1 larger than k and let it be k'. Then $k \to T$ is given as the SSYT with first row T_1 except that k' has been replaced by k and the remaining rows are given by the insertion $T_{2\sim} \to k'$.

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Let $w = 34112312$.

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3

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3	4

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3	
1	4

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3	4
1	1

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3	4	
1	1	2

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3	4		
1	1	2	3

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3			
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The Schensted algorithm is a piece of a well-known algorithm: Robinson-Schensted-Kunuth algorithm (RKS).

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3	4		
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T is katabolizable with respect to $\lambda^{(*)}$ if

- T contains a subtableau of shape and weight $\lambda^{(1)}$.
- $\mathrm{RKS}(v'u)$ is $(\lambda^{(2)}, \ldots, \lambda^{(r)})$ katabolizable with labels shifted by r.

Katabolizable Tableau

3			
2	2	3	
1	1	1	4

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4	
3	3

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The $\mathbb{K}^{\to k}$-Operator

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Definition 2.6

Let $\mathbb{K}^{\to k}$ be the operator on a set of tableaux with partition weight λ which acts by removing all tableau that are not katabolizable with respect to the k-split of λ and keeping all others.

The $\mathbb{K}^{\rightarrow k}$-Operator

Consider the set

3		
2	4	
1	1	5

3		
2	5	
1	1	4

4		
2	3	
1	1	5

4		
2	5	
1	1	3

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3		
2	4	
1	1	5

3		
2	5	
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4		
2	3	
1	1	5

4		
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1	1	3

The action of $\mathbb{K}^{\to 3}$ only keeps

3		
2	5	
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- $\mathbb{A}_{3211}^{(4)} = \begin{array}{|c|c|c|}\cline{1-1} 4 & \multicolumn{2}{c}{} \\ \cline{1-1} 3 & \multicolumn{2}{c}{} \\ \cline{1-2} 2 & 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|}\cline{1-1} 3 & \multicolumn{3}{c}{} \\ \cline{1-2} 2 & 2 & \multicolumn{2}{c}{} \\ \hline 1 & 1 & 1 & 4 \\ \hline \end{array}$

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- $s_{3211}^{(4)} = s_{3211} + ts_{421}$.

Outline

References

Thanks!

- T. Lam, L. Lapointe, J. Morse, A. Schilling, M. Shimozono, M. Zabrocki: *k*-Schur functions and affine Schubert calculus, 2012.