Quantum Groups Booklet

A Course in Quantum Groups

Henry Twiss
University of Minnesota

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Contents

1 Introduction to Quantum Groups

Quantum groups are roughly deformations of Lie groups (or more generally reductive algebraic groups).

Example 1.1 (A deformation of an algebra)**.** Let $k[X,Y]$ be a two variable polynomial ring over a field k with $XY = YX$. A deformation of this algebra is $k_q[X,Y] = \langle X, Y \mid qXY = YX \rangle$ where q is a fixed parameter[1] or a nonzero complex number.[2] As $q \to 1$ notice that $k_q[X,Y] \to k[X,Y]$.

Let's now look at an example of a complex Lie group.

Example 1.2 (Complex Lie group)**.** A prototypical example is

$$\mathrm{SL}(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{C}, ad - bc = 1 \right\}.$$

Note this group can be made into a differentiable manifold where the group operations are smooth.

A complex Lie group is a group which is also a complex differentiable manifold with some regularity conditions. In particular, a complex Lie group G is a complex differentiable manifold that is also a group such that the group maps (inversion and multiplication) are smooth with respect to the differential structure of the manifold.

We'd like to start off by understanding the representations of matrix groups G.[3] That is, understand the homomorphisms $\rho : G \to \mathrm{GL}(V) \cong \mathrm{GL}(n,\mathbb{C})$ where V is an n-dimensional complex vector space. To do this, to each matrix group (more generally to each Lie group) we associate its Lie algebra.[4]

Example 1.3 (A complex Lie algebra)**.** The Lie algebra associated to $\mathrm{SL}(2,\mathbb{C})$ is

$$\mathfrak{sl}(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{C}, a + d = 0 \right\},$$

that is the set of complex 2×2 traceless matrices. If $[X,Y] = XY - YX$ is the standard bracket operation, then this Lie algebra can be expressed as follows:

$$\langle E, F, H \mid [H,E] = 2E, [H,F] = -2F, [E,F] = H \rangle,$$

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[1] We also often think of q as a grading to extract finer information about $k[X,Y]$.

[2] We will see in the following why this is a good example of a deformation.

[3] For the concerned reader, all matrix groups are Lie groups, but the converse is not true.

[4] The Lie algebra arises from the Lie group by some exponential map.

where

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Returning to the general setting, let $\mathfrak{g}$ be the Lie algebra of G. If $\operatorname{Rep}(G)$ is the set of representations of G, then there is a natural map

$$\operatorname{Rep}(G) \to \operatorname{Rep}(\mathfrak{g}) \qquad \Pi \mapsto \pi(x) = \frac{d}{dt}\left(\Pi(e^{tx})\right)\Big|_{t=0}.$$

If the group has nice topological properties, then this map has an inverse.

Theorem 1.1. If G is connected and simply connected then there exists a correspondence between representations of G and representations of its Lie algebra.

We would like to pass from the Lie algebra to an associated object: the universal enveloping algebra of the Lie algebra. The universal enveloping algebra is a quotient of an object called the tensor algebra of the Lie algebra. Explicitly, the tensor algebra $T(\mathfrak{g})$ of $\mathfrak{g}$ is defined by

$$T(\mathfrak{g}) \coloneqq \bigoplus_{i=0}^{\infty} \mathfrak{g}^{\otimes i} = k \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus \cdots.$$

The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ is then defined by

$$\mathcal{U}(\mathfrak{g}) \coloneqq T(\mathfrak{g})/([x,y] - (x \otimes y) + (y \otimes x)),$$ [5]

where the ideal is generated by relations of the form $[x,y] - (x \otimes y) + (y \otimes x)$ for all $x, y \in \mathfrak{g}$.

Example 1.4 (Universal enveloping algebra)**.** The universal enveloping algebra of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ can be expressed as

$$\mathfrak{sl}(2,\mathbb{C}) = \langle e, f, h \mid he - eh = 2e, hf - fh = -2f, ef - fe = h \rangle.$$ [6]

It is a general fact that the universal enveloping algebra contains all the representations of the Lie algebra and has a center which acts by scalars on irreducible representations. The universal enveloping algebra is the algebra we would like to deform to get a quantum group.[7]

[5] Observe $x \otimes y$ and $y \otimes x$ are elements of $\mathfrak{g}^{\otimes 2}$ inside $T(\mathfrak{g})$.

[6] We are suppressing tensor notation in the relations.

[7] A quantum group is not a group in the algebraic sense, it is an algebra.

Example 1.5 (A quantum group)**.** The deformation of the universal enveloping algebra of $\mathfrak{sl}(2,\mathbb{C})$ is an example. Explicitly,

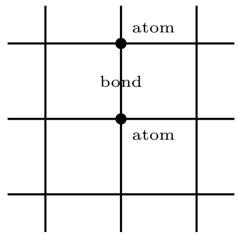
$$\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C})) = \langle E, F, K, K^{-1} \mid {}_{KFK^{-1}=q^{-2}F, [E,F]=EF-FE=(K-K^{-1})/(q-q^{-1})}^{KK^{-1}=1, K^{-1}K=1, KEK^{-1}=q^2E,} \rangle.^{8}$$

History of quantum groups: The term was coined by V. Drinfeld in his 1986 ICM lecture of the same title. He was attempting to formalize aspects of mathematical physics - exactly solvable lattice models and the quantum inverse scattering method. There was a tool known as the quantum Yang-Baxter equation which, in certain cases, could be used to solve the lattice model. Drinfeld realized that certain non-commutative Hopf algebras would produce quantum Yang-Baxter equations. These particular Hopf algebras are precisely quantum groups.

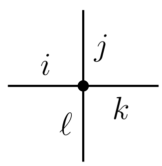
[8] It is not apparent from this form that setting $q = 1$ yields the universal enveloping algebra for $\mathfrak{sl}(2,\mathbb{C})$. We could write the quantum group in a different way to make this more apparent at the sacrifice of adding relations. We will also eventually answer the question of why this example of a quantum group is natural.

2 Partition Functions and Lattice Models

We're going to start with an application of quantum groups. The application comes from statistical mechanics: interpret macroscopic behavior of a system from microscopic interactions (say of atoms).[9] Our first setting is integrable 2-dimensional lattice models.[10]



These models look like rectangular grids of finite size, where we think the vertices as atoms and each vertex only interacts with its closest neighbors (i.e., only those vertices which share an edge). We think of the edges as bonds between atoms and we decorate them with elements of a finite set of size m to indicate various possible interactions (here we denote bonds as i, j, k, and ℓ).

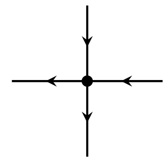


We also require a natural compatibility condition that decorations of adjacent vertices have to match, i.e., no edge gets two different decorations. Let's provide an example of an integrable 2-dimensional lattice model that we will use throughout the end of this discussion.

Example 2.1 (6-vertex model)**.** We will decorate every edge with an arrow either pointing in or out of a vertex. We require that every vertex has two adjacent edges pointing in and two adjacent edges pointing out. Disregarding adjacent vertices, there are $\binom{4}{2} = 6$ ways to decorate the edges of any vertex, and this is why we call it the 6-vertex model. For example, a vertex may have the following decoration:

[9]Be skeptical in general, but in specific cases this works.

[10]In the following, the notation 2-dimensional lattice models will be clear, but the notion of "integrable" will not be explained.



This example does arise from physics. Ice has a crystalline structure in which oxygen atoms (vertices) arrange themselves in a lattice and hydrogen atoms are closer to one oxygen atom in the lattice than its neighbors (bonds).

Given any vertex, there is a function $v \mapsto E_{i,j}^{k,\ell}(v)$ which sends v to the energy at v depending on the decorations (i, j, k, and ℓ). This function usually depends on the location of v in the grid, but we will assume otherwise. The goal is to infer global information from local functions. For example, we like to compute quantities such as the total energy

$$\sum_{\substack{\text{admissible}\\ \text{configurations}}} \sum_{\substack{v \text{ in}\\ \text{grid}}} E_{i,j}^{k,\ell}(v),$$

where the sum over admissible configurations means summing over all possible ways to decorate the lattice.[11] The probability that atoms arrange themselves into these configurations is inversely proportional to the energy.[12] Precisely, the probability is given by $e^{-\beta \cdot \text{Energy(state)}}$ where $\beta = 1/kT$, k is Boltzmann's constant, and T is temperature.

We would like an algebraic interpretation of the total probability (in this setting it's often called the partition function[13]). It is defined as

$$Z := \sum_{\substack{\text{admissible}\\ \text{configurations}}} e^{-\beta \cdot \text{Energy(state)}} = \sum_{\substack{\text{admissible}\\ \text{configurations}}} \prod_{\substack{v \text{ in}\\ \text{grid}}} e^{-\beta E_{i,j}^{k,\ell}(v)} = \sum_{\substack{\text{admissible}\\ \text{configurations}}} \prod_{\substack{v \text{ in}\\ \text{grid}}} R_{i,j}^{k,\ell}(v),$$

where $R_{i,j}^{k,\ell}(v) := e^{-\beta E_{i,j}^{k,\ell}(v)}$ is the Boltzmann weight of v.

Let us give an example of why it's important to compute Z.

Example 2.2. If Q is a physical quantity then the average value of Q is defined by

$$\langle Q \rangle := \frac{\sum_{\substack{\text{admissible}\\ \text{configurations}}} Q e^{-\beta \cdot \text{Energy(state)}}}{Z}.$$

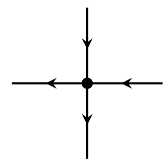
If $Q = E$, the energy of a system, then it's well-known that $\langle E \rangle = kT^2 \cdot \frac{\partial}{\partial T} \ln Z$. So, if we known Z we can compute $\langle E \rangle$ which is global information of a system.

[11] We often say a lattice with every edge decorated is an admissible configuration of the lattice.

[12] This is due to atoms arranging themselves in the lowest possible energy states.

[13] This has nothing to do with integer partitions.

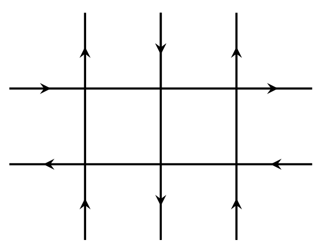
If Z can be explicitly computed then the model is said to be exactly solvable. Let's now give an example of Boltzmann weights. We will write directions such as (NE, SE, etc.) to stand for a vertex with its surrounding edges decorated such that the directions indicate which edges point inward. For example we would write NE for



Example 2.3 (Boltzmann weights)**.** Consider the following assignment of Boltzmann weights

Directions	SE	NW	NE	SW	NS	EW
$R_{i,j}^{k,\ell}$	1	1	λ	λ	$1-q\lambda$	$1-q^{-1}\lambda$

Here λ is an arbitrary complex parameter and q is a nonzero parameter. and consider the following 2-dimensional lattice model[14]

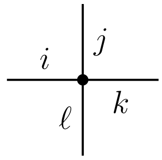


Computing Z for the diagram above means we need to find all possible fillings satisfying two arrows pointing in and two arrows pointing out at each vertex, calculating the Boltzmann weight of each, and then summing them.[15]

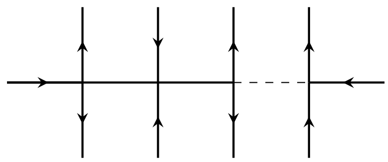
This leads us to understanding an algebraic interpretation of Z which is usually easier to compute. Given m decorations, we define an abstract m-dimensional complex vector space V generated by $v_1, \ldots, v_m$. We encode the $R_{i,j}^{k,\ell}$ as matrix coefficients of an endomorphism of $V \otimes V$. We call this endomorphism R. For example, in the 6-vertex model $\dim(V) = 2$ and R is a 4×4 matrix (because $\dim(V \otimes V) = 4$) with 6 nonzero entries where each nonzero entry corresponds to a Boltzmann weight. This suggests the following interpretation: when you see a vertex

[14] This is not an admissible configuration since every edge is not decorated.

[15] This gets messy very quickly.



think of it as $v_i \otimes v_j \mapsto R_{i,j}^{k,\ell}(v) \cdot (v_k \otimes v_\ell)$ where the mapping is given by R. With this interpretation in mind we can solve for Z one row at a time. Consider the arbitrary 2-dimensional lattice model with N columns.



We can think of the left most edge as an element of V and the top set of edges as an element of $V^{\otimes N}$. If T is a row of the partition function, then we can interpret T as belonging to $\text{End}(V \otimes (V^{\otimes N}))$. If we record elements of $V \otimes (V^{\otimes N})$ as $v_0 \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_N$, then we have the following theorem:

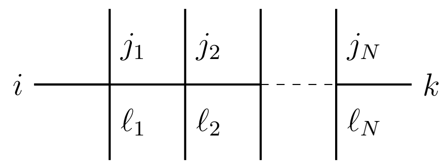
Theorem 2.1.

$$T = R_{(0,1)} R_{(0,2)} R_{(0,3)} \cdots R_{(0,N)}$$

where $R_{(i,j)}$ means apply R to $v_i \otimes v_j$ and apply the identity everywhere else.

3 One-Row Partition Functions

Consider the more general 2-dimensional lattice model with N columns:



Let $T \in \mathrm{End}(V \otimes (V^{\otimes N}))$ be the associated $(m^{N+1}) \times (m^{N+1})$ matrix (called a transfer matrix). In particular,

$$v_i \otimes (v_{j_1} \otimes \cdots \otimes v_{j_N}) \mapsto \cdots + T^{k,\ell_s}_{i,j_s} \cdot v_k \otimes (v_{\ell_1} \otimes \cdots \otimes v_{\ell_N}) + \cdots .$$

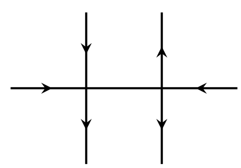
Then we have the following:

Theorem 3.1.

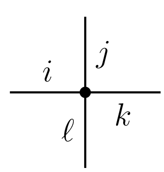
$$T = R_{(0,1)} R_{(0,2)} \cdots R_{(0,N)}.$$

Lets see an example of this theorem in the 6-vertex model for a one-row partition function.

Example 3.1 (Computing T in the 6-vertex model)**.** We would like to compute T of the following 2-dimensional lattice model:



There is only one admissible configuration so our sum only has one term, and by the definition of Z (in this case $Z = T$), we have $T = \mathrm{wt(NW)wt(EW)}$. If we label the orientations $\uparrow$ and $\leftarrow$ with a $+$ and the orientations $\downarrow$ and $\rightarrow$ with a $-$, then any quadruple of $+$ and $-$ symbols corresponds to a decoration of



where we start decorating at edge ℓ and move clockwise. If we order the rows and columns of R by $++, +-, -+, --$, then we have

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 1-q\lambda & 0 \\ 0 & 1-q^{-1}\lambda & \lambda & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.^{16}$$

Therefore $T = \mathrm{wt}(\mathrm{NW})\mathrm{wt}(\mathrm{EW}) = 1 \cdot 1 - q^{-1}\lambda = 1 - q^{-1}\lambda$, and we claim the coefficient $T^{+,-,-}_{-,-,+}$ is precisely this value in $T \in \mathrm{End}(V \otimes (V \otimes V))$. The ordering of the basis for $V \otimes V \otimes V$ is $+++, +-+, -++, --+, ++-, +--, -+-, --$ since it respects the ordering of the $++, +-, -+, --$. That is, if we throw away the last $\pm$ symbol we get the ordering of the basis for $V \otimes V$ and the last $\pm$ symbol is a $+$ before a $-$. The 4-th element of the order is $--+$ is the 6-th is $+--$ so we are computing the $(4, 6)$-entry of T. Now $R_{(0,1)}$ acts by R on the first two copies of V in $V \otimes (V \otimes V)$ and the identity on the third, and under the same basis we may write

$$R_{(0,1)} = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}.^{17}$$

On the other hand, $R_{(0,2)}$ acts by R on the first and third copy of V and the identity on the second. So to obtain $R_{(0,2)}$ we take $R_{(0,1)}$ and apply the base change which interchanges the last two copies of $V \otimes V \otimes V$. The result is:

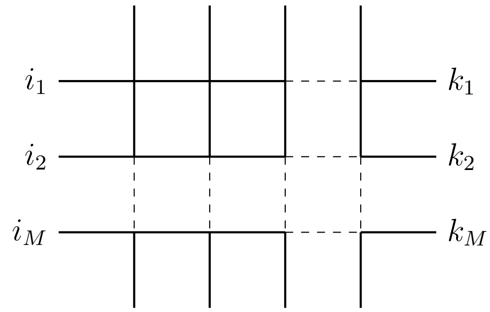
$$R_{(0,2)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 1-q^{-1}\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 1-q^{-1}\lambda & 0 & 0 \\ 0 & 0 & 1-q\lambda & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-q\lambda & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

If we multiply the 4-th row of $R_{(0,1)}$ by the 6-th column of $R_{(0,2)}$ we get $1 - q^{-1}\lambda$ as claimed.

For a generalized 2-dimensional lattice model,

[16] We implicitly chose an ordering of the basis, but all claims are independent of this choice.

[17] Recall that the tensor product of matrices is the Kronecker product.



where $\underline{j}$ and $\underline{\ell}$ will stand for the N-tuple of decorations of edges on the top and bottom row respectively and $\underline{r}^{(1)}, \ldots, \underline{r}^{(M-1)}$ are the N-tuples of decorations of edges for the intermediary rows. The partition function Z may be described as

$$Z = \sum_{\underline{r}^{(M-1)}} \sum_{\underline{r}^{(1)}} T_{i_1,\underline{j}}^{k_1 \underline{r}^{(1)}} \cdot T_{i_2,\underline{r}^{(1)}}^{k_2,\underline{r}^{(2)}} \cdots T_{i_M,\underline{r}^{(M-1)}}^{k_M \underline{\ell}}.^{18}$$

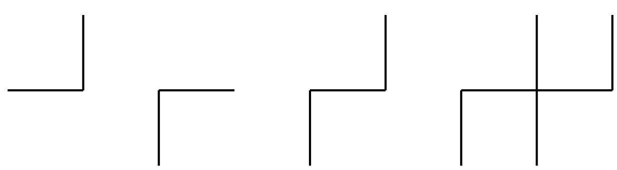
If we assume $i_s = k_s$ for $1 \leq s \leq M$, then we may identify them so our grid becomes a cylinder[19] and we may write the partition function as

$$Z = \left(\mathrm{trace}_V(T)^M\right)_{\underline{j}}^{\underline{\ell}}.$$

where $\mathrm{trace}_V(T)^M \in \mathrm{End}(V^{\otimes N})$ is the partial trace of T. Moreover, if we also have $\underline{j} = \underline{\ell}$ then we may make identifications such that our grid is a torus[20] and our partition function becomes

$$Z = \mathrm{trace}_{V^{\otimes N}} \left(\mathrm{trace}_V(T)^M\right).^{21}$$

Assuming toroidal boundary conditions, if we can find the largest eigenvalue of $\mathrm{trace}_V(T)$, let's call it k_N, then as $M \to \infty$ $Z \sim k_N^M$ asymptotically. With cylindrical boundary conditions we are able to model states as non-intersecting lattice paths as follows: whenever we see a $\uparrow$ or $\to$ we put a path and otherwise we do nothing. Some examples of states are pictured below.



[18]This is almost like matrix multiplication.
[19]This case is referred to as cylindrical boundary conditions.
[20]This is referred to as toroidal boundary conditions.
[21]The exponent inside the parentheses is just indicating matrix multiplication M times.

Toroidal boundary conditions imply a path never ends which further implies conservation of up arrows in rows. Hence $\text{trace}_V(T)$ breaks up into blocks according to the number of up arrows (i.e., paths) from the bottom row. This lets us determine the eigenvalues.[22] We would like to understand a different method of computing the eigenvalues by using quantum groups.

[22]The interested reader could consult Chapter 8 of Baxter's book *Exactly Solved Models in Statistical Mechanics.*

4 The Quantum Yang-Baxter Equation

Recall that for toroidal boundary conditions,

$$Z = \text{trace}_{V^{\otimes N}} \left(\text{trace}_V(T)^M \right).$$

Set $A = \text{trace}_V(T)$, we also call A a transfer matrix.[23] We want to analyze A in particular settings. In the case of the 6-vertex model with toroidal boundary conditions we can completely understand $A \in \text{End}(V \otimes V)$. Here is a sketch of the argument originally developed by Lieb in 1967 and generalized by Sutherland in the same year:

1. Model states as lattice paths for transfer matrices.
2. This splits A into blocks according to the number of paths.[24]
3. Use clever trick to diagonalize blocks.[25]

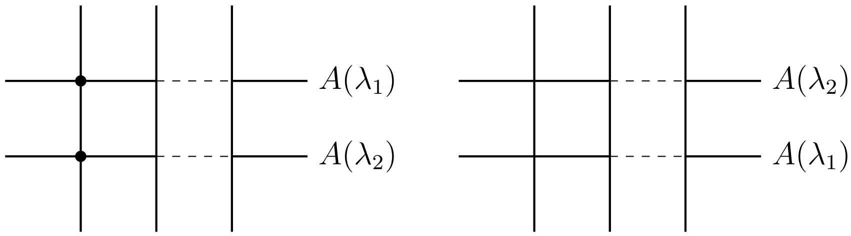
As a result, we may write $A = PA_{\text{diag}}P^{-1}$ where P is the matrix of eigenvectors and A_{diag} is the matrix of distinct eigenvalues.

In general, the 6-vertex model has six free parameters (the weights). It is a reasonable assumption that the weights are symmetric upon reversal of arrows (e.g., upon reversal of arrows NS $\mapsto$ EW). Assuming this additional constraint, our model is reduced to three free parameters, and physicists call this setting the three parameter field-free model. In the field-free setting it so happens that P is independent of one of the three parameters. Label this independent parameter λ_i if we are considering the transfer matrix associated to the i-th row of our lattice (we do this purely to distinguish between transfer matrices in different rows). Write $A_{\text{diag}}(\lambda_1)$ and $A_{\text{diag}}(\lambda_2)$ to emphasises that A_{diag} depends on this parameter (and similarly for A). These matrices are diagonal so they commute implying $A(\lambda_1)A(\lambda_2) = A(\lambda_2)A(\lambda_1)$ (i.e., transfer matrices commute). Pictorially, for any choice of $\underline{j}$ and $\underline{\ell}$, the partition functions corresponding to the 2-dimensional lattice models below are the same under the constraints mentioned above:

[23] Notice that A also depends on what row of our lattice we are considering since this tells us which copy of V to collapse when taking the trace.

[24] This is where the toroidal boundary conditions are used critically.

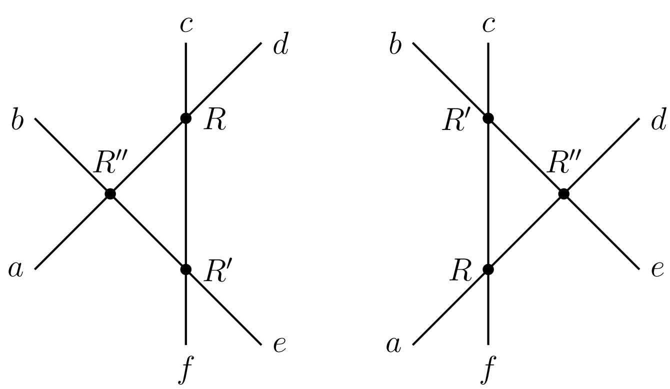
[25] The Bethe Ansatz method is used here, and it turns out that the eigenvalues of A are all distinct.



This fact should come as a surprise because the weights corresponding to the vertices emphasised in the left-most lattice are affected by different boundary conditions, but are invariant under interchanging rows.

A natural question to ask is if this process is reversible. That is, can one get from commuting transfer matrices to a determination of all the eigenvalues and eigenvectors of A? The answer is surprisingly yes under arbitrary boundary conditions.[26] The goal now is to determine sufficient conditions under which transfer matrices commute. The following theorem of Yang and Baxter accomplishes this:

Theorem 4.1 (Quantum Yang-Baxter equation)**.** A sufficient condition for transfer matrices to commute is an $R'' \in \mathrm{End}(V \otimes V)$ such that the partition functions corresponding to the 2-dimensional lattice models below are equal

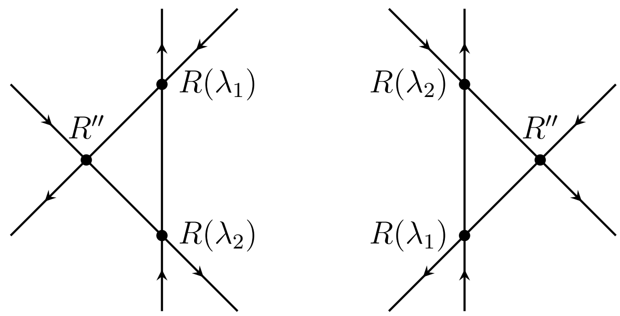


for any choice of decorations a, b, c, d, e, and f.

In the 6-vertex model, there are two decorations for each edge, so there are 2^6 equations and R'' is a 4×4 matrix with 6 unknowns. Let's see an example in this setting.

[26] The interested reader can consult section 9.5 of Baxter's book *Exactly Solved Models in Statistical Mechanics* where he answers this question in the setting of the 6-vertex model.

Example 4.1 (Quantum Yang-Baxter equation in the 6-vertex model)**.** We want to find a matrix of weights R'' such that the partition functions corresponding to the 2-dimensional lattice models



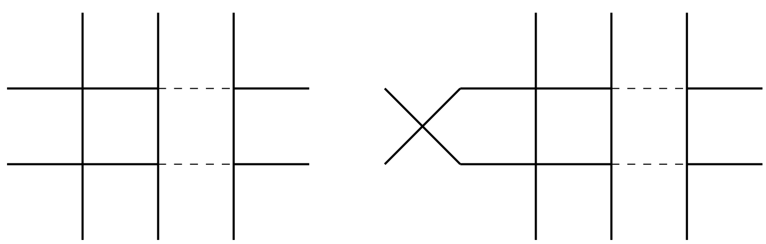
are equal. If we rotate the cross (only) in the left 2-dimensional lattice $\pi/4$ radians counter-clockwise, we may express its partition function as

$$R''(\mathrm{NW})\mathrm{SE}(\lambda_1)\mathrm{SW}(\lambda_2) + R''(\mathrm{EW})\mathrm{EW}(\lambda_1)\mathrm{NS}(\lambda_2),$$

since there are only two admissible configurations.

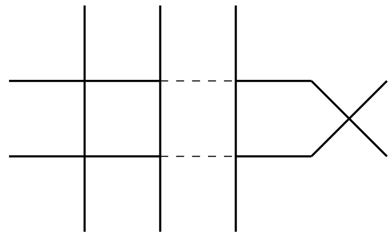
Why might the quantum Yang-Baxter equation guarantee transfer matrices commute? The argument is as follows.

Proof sketch. For simplicity, assume cylindrical boundary conditions, and consider the 2-dimensional lattice models (we're suppressing decorations here):



Denote the partition function of the left model by Z_l, the partition function with interchanged rows by Z_l', the partition function of the right model by Z_r, and let wt stand for the weight of the cross in the right model. Then $Z_r = \mathrm{wt} \cdot Z_l$. By repeatedly applying the quantum Yang-Baxter equation, the partition function of the right model is invariant under shifting the cross across columns given we interchange the weights of the column we pass.[27] Moving the cross across the entire lattice produces the model

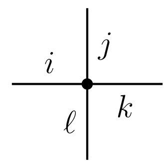
[27]This type of argument is sometimes called a "train argument".



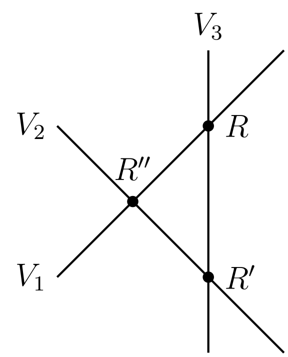
where the vertical rows are interchanged and $Z_r = Z_l' \cdot \mathrm{wt}$. Hence $\mathrm{wt} \cdot Z_l = Z_l' \cdot \mathrm{wt}$ implying the transfer matrices commute. $\square$

5 Solutions to the Quantum Yang-Baxter Equation

Recall that when we see a vertex



we think of it as $v_i \otimes v_j \mapsto R_{i,j}^{k,\ell}(v) \cdot (v_k \otimes v_\ell)$ where the mapping is given by R. In other words, we associate an element $R \in \mathrm{End}(V \otimes V)$ to this picture. Suppressing decorations and labeling lines, consider



where V_1, V_2, and V_3 are vector spaces labeling the lines. We can think of this picture as an element of $\mathrm{End}(V_1 \otimes V_2 \otimes V_3)$ where the segments of the edges closest to V_1, V_2, and V_3 are inputs and when we pass a vertex by moving left the inputs are acted upon by matrices R'', R, and R' respectively. Moreover, the matrices only act on the vector spaces associated to the lines which intersect at their vertex (so R'' acts on V_1 and V_2 and acts as the identity on V_3). The quantum Yang-Baxter equation then reads

$$R''RR' = R'RR''.$$

Notice that this is a purely algebraic statement, so we have an algebraic interpretation of the quantum Yang-Baxter equation.[28]

We now understand that solutions to the quantum Yang-Baxter equation gives rise to commuting transfer matrices (by a train argument) and these commuting

[28]The interested reader can find an algebraic proof of the quantum Yang-Baxter equation in 7.5 of Chari and Pressley's *A Guide to Quantum Groups*.

transfer matrices solve[29] the partition function Z by a method of Baxter. We would now like to answer the following questions about the quantum Yang-Baxter equation:

1. Are there solutions to the quantum Yang-Baxter equation?
2. If there are solutions, can we provide a source for lots of solutions?
3. What does the quantum Yang-Baxter equation have to do with quantum groups?

The answer to the first question is yes and we would like to illustrate it with an example in the 6-vertex model.

Example 5.1 (Solutions to quantum Yang-Baxter equation for the 6-vertex model)**.** In our running example we have $R(\lambda_1, q)$ and $R(\lambda_2, q)$. The solution to the quantum Yang-Baxter Equation is also of this form: $R'' = R''(\lambda_3, q)$ where λ_3 satisfies the follow equation

$$\lambda_3 - \lambda_1 + \lambda_2 + \lambda_1\lambda_2\lambda_3 = (q + q^{-1})\lambda_3\lambda_2.$$

Label the weights of SW, NW, NE, SE, EW, and NS by a_1, b_1, a_2, b_2, c_1, and c_2 (in the field-free setting $a_1 = a_2$, $b_1 = b_2$, and $c_1 = c_2$) and define invariants

$$\Delta_1 \coloneqq \frac{a_1a_2 + b_1b_2 - c_1c_2}{2a_1b_1} \qquad \Delta_2 \coloneqq \frac{a_1a_2 + b_1b_2 - c_1c_2}{2a_2b_2}$$

(in the field-free setting $\Delta_1 = \Delta_2$). Then we have a theorem:

Theorem 5.1 (Brubaker-Bump-Friedberg)**.** In the 6-vertex model, there exists a solution R'' to the quantum Yang-Baxter equation if and only if $\Delta_1(R) = \Delta_1(R')$ and $\Delta_2(R) = \Delta_2(R')$. In particular, R'' can be described in terms of R and R'.

If we are not in the 6-vertex model there is still a general recipe for obtaining solutions to the quantum Yang-Baxter equation. We outline this recipe and in it answer the second and third questions. The following is technical. In the general setting, representations of quantum groups give rise to solutions of the quantum Yang-Baxter equation. Quantum groups themselves are special examples of Hopf algebras. To be precise, quantum groups are quasi-triangular Hopf algebras. That is, a pair $(H, \mathcal{R})$ consisting of a Hopf algebra H and a an element $\mathcal{R} \in H \otimes H$ obeying nice properties including an "arbitrary quantum Yang-Baxter equation" $\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$. To obtain solutions of the quantum Yang-Baxter equation, one takes a representation (ρ, V) of H and a quantum group $(H, \mathcal{R})$. It so happens that $(\rho \otimes \rho)(\mathcal{R})$ give

[29] By solve we mean described in closed form using familiar functions.

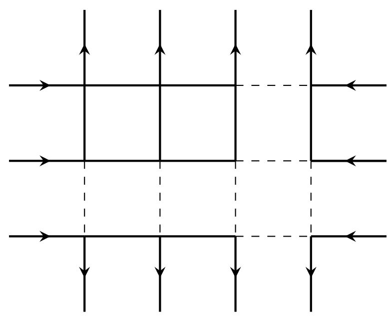
an "honest" matrix solution to the quantum Yang-Baxter equation. In particular, the quantum group $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ gives rise to solutions to the quantum Yang-Baxter equation for the 6-vertex model.

We'd now like to give a simple in-depth example of computing the entire partition function.

Example 5.2 (Ice)**.** In ice, the energy of every oxygen atom is the same. Therefore we my assume all weights are 1. This means we may express our partition function as

$$Z = \sum_{\substack{\text{admissible} \\ \text{configurations}}} 1.$$

That is, Z counts the fillings of the lattice given boundary conditions. Let $M = N$ and consider the 2-dimensional lattice model



The admissible fillings of this lattice are in bijections with $N \times N$ alternating sign matrices (matrices with entries in $0, 1, -1$ such that nonzero entries in rows and columns alternate $1, -1, 1 \dots$ and entries in rows and columns sum to 1). Given a vertex in an admissible configuration of the lattice, assign a 0 to the corresponding entry in a $N \times N$ matrix unless the weight is EW in which case assign a 1 or NS in which case assign a -1. It is a theorem of Zeilberger, Kuperberg, Stroganov, and Okada (all independently) that the number of $N \times N$ alternating sign matrices is given by the formula

$$\frac{1!4!7! \cdots (3N-2)!}{N!(N+1)! \cdots (2N-1)!}.$$

Zeilberger's proof establishes shows these matrices and another combinatorical object are equinumerous but he does not establish a bijection. Kuperberg's, Stroganov's, and Okada's proof use quantum groups, which are more elegant.

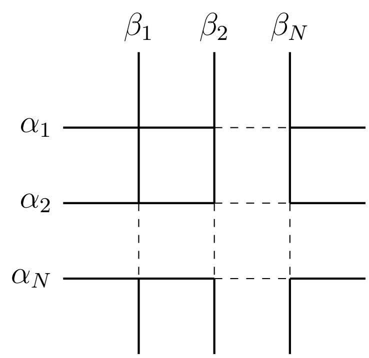
6 Stroganov's and Okada's Proof and Generalizations

We would like to discuss the proof strategy that Stroganov and Okada used to count the number of $N \times N$ alternating sign matrices[30].

In the field-free setting define Boltzmann weights

$$\begin{aligned} a &:= qx - q^{-1}x^{-1}, \\ b &:= x - x^{-1}, \\ c &:= q - q^{-1}, \end{aligned}$$

where x and q are free parameters. Now let $\alpha_1, \ldots, \alpha_N$ and $\beta_1, \ldots, \beta_N$ be free parameters and label the rows and columns of the following 2-dimensional lattice model with them as pictured below (notice the lattice has N rows and N columns):



Assign weights as in the field-free setting (after decorating edges), but set $x = \alpha_i/\beta_j$ if the vertex is in the row α_i and column β_j. Stroganov and Okada realized that if we set $q = e^{i\pi/3}$, $\underline{\alpha} = (q^{-1}, \ldots, q^{-1})$ and $\underline{\beta} = (1, \ldots, 1)$, then $Z_N(\underline{\alpha}, \underline{\beta})$ is a complex multiple of the number of $N \times N$ alternating sign matrices. So, we need to compute $Z_N(\underline{\alpha}, \underline{\beta})$. It is a claim that $Z_N(\underline{\alpha}, \underline{\beta})$ is characterized by the following properties

1. $Z_N(\underline{\alpha}, \underline{\beta})$ is symmetric in the entries of $\underline{\alpha}$.
2. $Z_N(\underline{\alpha}, \underline{\beta})$ is symmetric in the entries of $\underline{\beta}$.
3. For each $1 \leq i \leq N$, $\alpha_i^{N-1} Z_N(\underline{\alpha}, \underline{\beta})$ is a polynomial of degree at most $N-1$ in α_i^2.

[30] The interested reader can consult *Six-Vertex, Loop, and Tiling Models: Integrability and Combinatorics* by Zinn-Justin for a deeper discussion.

4. For each $1 \leq i \leq N$, $\beta_i^{N-1} Z_N(\underline{\alpha}, \underline{\beta})$ is a polynomial of degree at most $N-1$ in β_i^2.

5. There is a relationship between $Z_N(\underline{\alpha}, \underline{\beta})$ and $Z_{N-1}(\alpha_2, \ldots, \alpha_N, \beta_2, \ldots, \beta_N)$ if $\alpha_1 = \beta_1$. Explicitly,

$$Z_N(\underline{\alpha}, \underline{\beta}) = (q - q^{-1}) \prod_{i=2}^{N} \left(q\frac{\alpha_1}{\alpha_i} - q^{-1}\frac{\alpha_i}{\alpha_1} \right) \prod_{j=2}^{N} \left(q\frac{\beta_j}{\alpha_1} - q^{-1}\frac{\alpha_1}{\beta_j} \right) \cdot Z_{N-1}(\alpha_2, \ldots, \alpha_N; \beta_2, \ldots, \beta_N).$$

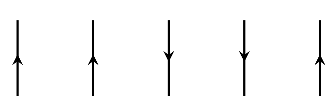
Property (1) is proved as follows: Show $Z_N(\underline{\alpha}, \underline{\beta})$ has an associated solution to the quantum Yang-Baxter equation by showing $\Delta(\underline{x}, q)$ is independent of x.[31] Then use a train argument to illustrate that $Z_N(\underline{\alpha}, \underline{\beta})$ is invariant under interchanging rows of the model (i.e., is symmetric in the entries of $\underline{\alpha}$).[32] (2) is proved by rotating the lattice and using an analogous argument. (3) and (4) we encourage the reader to prove themselves. We exclude the proof of (5).

Izergin found an expression for a function with these properties by using q-binomial coefficients and determinants in them. When we set $q = e^{i\pi/3}$, $\underline{\alpha} = (q^{-1}, \ldots, q^{-1})$, and $\underline{\beta} = (1, \ldots, 1)$, the result is easily seen to be

$$\frac{1!4!7! \cdots (3N-2)!}{N!(N+1)! \cdots (2N-1)!}.$$

One can use similar techniques to evaluate a more general class of 6-vertex 2-dimensional lattice models with specified boundary conditions. Say the lattice has r rows and ℓ columns. The top boundary conditions are a sequence of up and down arrows and we encode this sequence by an integer partition λ with nonnegative distinct parts such that $\lambda + \rho$ has distinct parts where $\rho = (r-1, r-2, \ldots, 1, 0)$. Given $\lambda + \rho$, we put an up arrow at column i if i is in a part of $\lambda + \rho$ and put a down arrow otherwise. Let's see an example.

Example 6.1 (Specifying boundary conditions using integer partitions)**.** If $\lambda = (2, 2, 0)$ then $\rho = (2, 1, 0)$ and $\lambda + \rho = (4, 3, 0)$. The top boundary conditions are then given by the the sequence of arrows



[31]Notice we're using the result of Brubaker, Bump, and Friedberg.

[32]We don't assume toroidal boundary conditions here so we need to do a little more work to show the weights we interchange are the same as we apply the train argument.

We encode the bottom boundary analogously. Given a matrix with top boundary $\lambda + \rho_r$, bottom boundary $\mu + \rho_\ell$, left boundary all right right arrows, and right boundary all left arrows, we have a theorem of Brubaker, Bump, and Friedberg:

Theorem 6.1 (Brubaker-Bump-Friedberg)**.** If $\Delta = 0$, then we may write the partition function as

$$Z_{\lambda+\rho_r/\mu+\rho_\ell} = Z_{\rho_r/\rho_\ell} \cdot S_{\lambda/\mu}\left(\frac{b_2^{(1)}}{a_1^{(1)}}, \dots, \frac{b_2^{(r-\ell)}}{a_1^{(r-\ell)}}\right),$$

where Z_{ρ_r/ρ_ℓ} is a computable partition function depending on ρ_r and ρ_ℓ, $S_{\lambda/\mu}$ is a skew Schur polynomial depending on λ and μ, and the $a_1^{(i)}$'s and $b_2^{(j)}$'s are weights.

7 Relations Among Symmetric Functions

In the previous section we discovered that $Z_N(\underline{\alpha}, \underline{\beta})$ was symmetric in the entries of $\underline{\alpha}$ and $\underline{\beta}$. More generally, we would like to know which symmetric functions are representable as partition functions of a 2-dimensional lattice model (not necessarily in the 6-vertex setting). A natural question to ask is why we care about which symmetric functions are representable as partition functions. The punchline is that partition functions satisfy many functorial properties so we can often prove a lot of identities about symmetric functions if we represent them as partition functions.

Recall that given a partition λ with r nonnegative parts and $\rho = (r-1, r-2, \ldots, 1, 0)$, $\lambda+\rho$ is a partition with distinct parts which encodes boundary conditions for a 2-dimensional lattice model. In the 6-vertex model, if we declare all the left boundary conditions to point to the right, all the right boundary conditions to point to the left, and all the bottom boundary conditions to point down, then we have an assignment $\lambda \mapsto Z_\lambda$ with with weights given by

$$\begin{aligned} a_1 &= 1 & a_2 &= x_i, \\ b_1 &= 0 & b_2 &= x_i, \\ c_1 &= x_i & c_2 &= 1, \end{aligned}$$

where i is an index for the row number. In this setting we have the following theorem:

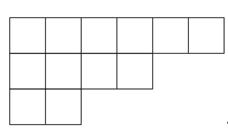
Theorem 7.1. Let $\underline{x} = (x_1, \ldots, x_r) \in \mathbb{C}^r$. Then

$$Z_\lambda(\underline{x}) = \underline{x}^\rho \cdot S_\lambda(\underline{x}),$$

where $\underline{x}^\rho = (x_1^{r-1}, x_2^{r-2}, \ldots, x_{r-1}, x_r^0)$ and $S_\lambda(\underline{x})$ is the Schur polynomial corresponding to λ.

What do we mean when we say $S_\lambda(\underline{x})$ is the Schur polynomial corresponding to λ? If we have a Young diagram corresponding to the partition λ, we may fill the boxes of the diagram with elements from the alphabet $\{1, 2, \ldots, r\}$ to make it a semi-standard Young tableaux.[33] If T is a semi-standard Young tableaux, then the weight $\mathrm{wt}(T)$ of T is defined to by the r-tuple where the i-th entry consists of the number of boxes filled with letter i in the alphabet. Let's see an example.

Example 7.1 (Semi-standard Young tableaux)**.** The Young diagram corresponding to $\lambda = (6, 4, 2)$ is



.

[33] This means the row entries weakly increase and the column entries strictly increase.

Our alphabet is $\{1, 2, 3\}$ and a filling of this Young diagram which produces a semi-standard Young tableaux T is

$$\begin{array}{|c|c|c|c|c|c|}
\hline
1 & 1 & 2 & 2 & 2 & 3 \\
\hline
2 & 2 & 3 & 3 \\
\cline{1-4}
3 & 3 \\
\cline{1-2}
\end{array} .$$

Then $\mathrm{wt}(T) = (2, 5, 5)$.

Letting $\mathrm{SSYT}(\lambda)$ stand for the set of all semi-standard Young tableaux with Young diagram corresponding to λ, the Schur polynomial $S_\lambda(\underline{x})$ corresponding to λ is defined by

$$S_\lambda(\underline{x}) := \sum_{T \in \mathrm{SSYT}(\lambda)} \underline{x}^{\mathrm{wt}(T)}.$$

There is an interesting inner product (called the Hall inner product[34]), and the Schur polynomials form an orthonormal basis with respect to this inner product. Moreover, in representation theory, irreducible and finite dimensional (necessarily polynomial) representations of $\mathrm{GL}_n(\mathbb{C})$ are indexed by partitions, and the characters of matrices with eigenvalues $x_1, \ldots, x_n$ are given by $S_\lambda(\underline{x})$. Pictorially,

$$\mathrm{GL}_n(\mathbb{C}) \xrightarrow{\rho_\lambda} \mathrm{End}(V) \xrightarrow{\mathrm{Tr}} \mathbb{C} \qquad \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_r \end{pmatrix} \mapsto S_\lambda(\underline{x}).$$

As an aside, there are some other interesting polynomials which have connections to symmetric polynomials. We will speak somewhat loosely in the following. The Macdonald polynomials $P_\lambda(\underline{x}; q, t)$ are polynomials dependent on two parameters q and t. If we set $q = 0$, then we obtain the Hall-Littlewood polynomials $P_\lambda(\underline{x}; t)$, and setting $t = 0$ gives the Schur polynomials $S_\lambda(\underline{x})$. If we take the Macdonald polynomials, set $t = q^a$ and perform a limiting procedure $q \to 1$, we obtain the Jack polynomials. One can define the so-called non-symmetric Macdonald polynomials $E_\alpha(\underline{x}; q, t)$ where α is a composition. If we "average over all permutations of α," we get the usual Macdonald polynomials $P_\lambda(\underline{x}; q, t)$.

In 2019, Berodin and Wheeler produced a 2-dimensional lattice model with quantum Yang-Baxter equations whose partition function was $E_\alpha(\underline{x}; q, t)$ with $\underline{x} \in \mathbb{C}^n$. The decorations on an edge come from the quantum group module $\mathcal{U}_q(\hat{\mathfrak{sl}}(n+1, \mathbb{C}))$.[35]

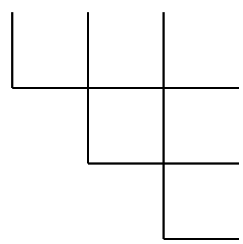
[34] It is a symmetric bilinear form on the ring of symmetric functions.

[35] Here $\hat{\mathfrak{sl}}(n+1, \mathbb{C})$ can be thought of as an affine version of $\mathfrak{sl}(n+1, \mathbb{C})$.

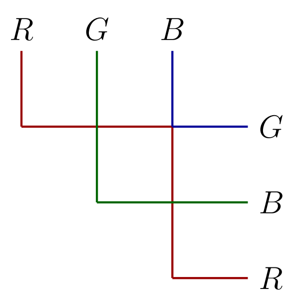
In particular, the decorating set is unbounded but it comes from the quantum group module above and so the associated quantum Yang-Baxter equation has a solution!

In fact, there is a more general recipe which takes a symmetric function and breaks it into non-symmetric pieces. If we assume toroidal boundary conditions recall that we can think of decorations (in the 6-vertex model) as paths. If we color the starting and ending points of the paths with the same set and permute them then the symmetric function breaks into pieces according to which permutations give rise to paths with the same start and end color. Let's see an example.

Example 7.2 (Breaking symmetric functions according to path colorings)**.** If we have the following path diagram



Then the *RGB* coloring below corresponds to a non-symmetric piece (notice the right labeling has been permuted):



The important fact here is that if a path emanates from a coloring at the top it needs to terminate at the corresponding coloring on the right.

There are two other interesting properties of Schur polynomials that we can deduce using lattice models. The first is that if we have a lattice model with top boundary conditions given by $\lambda + \rho$ and remove the top row this gives rise to a branching rule. In particular, we have the property

$$S_\lambda(\underline{x}) = \sum_\mu c_\mu \cdot S_\mu(\underline{x}),$$

where μ ranges over the partitions with one less part than λ and c_μ is a constant depending on μ. In the same setting we can also glue certain lattices together. If $\lambda+\rho$ are the top boundary conditions for a lattice model, let $\tilde{\lambda}+\rho$ be the boundary conditions such that if we flip the $\tilde{\lambda}+\rho$ upside down its bottom row decorations match with the decorations of $\lambda+\rho$. If Z denotes the partition function for the new lattice model then we have

$$Z = \sum_{\mu} Z_\mu(\underline{x}) Z_{\tilde{\mu}}(\underline{x}),$$

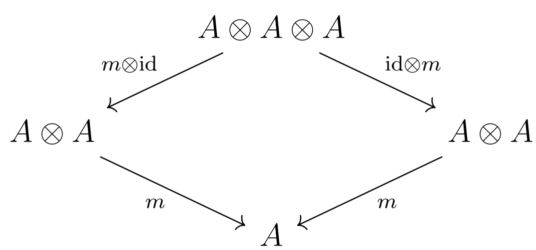
where μ ranges over all partitions corresponding to the gluing. This identity is called the dual Cauchy identity.

8 Algebras, Coalgebras, and Bialgebras

Let A be an algebra over k. In the remainder of these notes, by an algebra we always mean and associative algebra with unit. That is, a k-vector space A with an associative multiplication, by which we mean a k-linear map $m : A \otimes A \to A$, and a unit element 1_A with

$$m(a \otimes 1_A) = m(1_A \otimes a) = a.$$

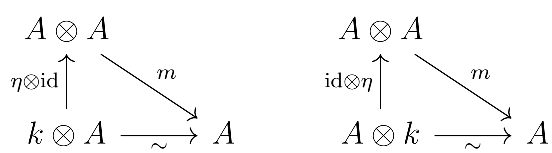
In terms of diagrams, associativity of the algebra means that the following diagram commutes:



Now for a little more about the unit 1_A. For any $a \in A$ there exists a map

$$\eta_a : k \to A \qquad \lambda \mapsto \lambda a.$$

In particular, $\eta_a(1) = a$. If there exists a unit $1_A \in A$, we have $\eta(1) := \eta_{1_A}(1) = 1_A$. In diagrams, A possessing a unit is equivalent to a k-linear map η such that the following diagrams commute



With this in mind we can identify the unit in the algebra with the unit in k. We will call the common unit 1. This also lets us embed the field k in A.

An algebra morphism (or algebra homomorphism) $f : A \to A'$ is a k-linear map such that $f \circ m = m' \circ (f \otimes f)$ and $f \circ \eta = \eta'$ where m' and η' are the multiplication and unit of A' respectively.

Example 8.1 (Algebras)**.** We have two primary examples of interest.

- Let k be a field and G be any finite group. Then the group algebra $k[G]$ is defined as the k-vector space with basis $\{e_g\}_{g \in G}$ and the multiplication is given by $m(e_g \otimes e_h) = e_{gh}$ where gh denotes multiplication in G. The unit is given by $\eta(k) = ke_1$ where 1 is the identity in G.

- Let V be a vector space. Then the tensor algebra $T(V) = \bigoplus_{i=0}^{\infty} V^{\otimes i}$ (which we have already encountered) can be made into an algebra. Let $T^i(V) = V^{\otimes i}$. Then there is a canonical isomorphism

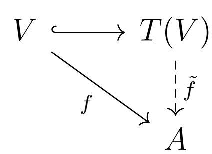
$$T^n(V) \otimes T^m(V) \mapsto T^{n+m}(V)$$

define by tensoring elements. We can extend this map linearly to a multiplication $m : T(V) \otimes T(V) \to T(V)$. In particular, given two elements in $T(V)$ we first distribute and then apply the canonical isomorphism above to each pair. For example

$$m((k + v_1) \otimes (v_1 \otimes v_2)) = (kv_1 \otimes v_2) + (v_1 \otimes v_2 \otimes v_3).$$

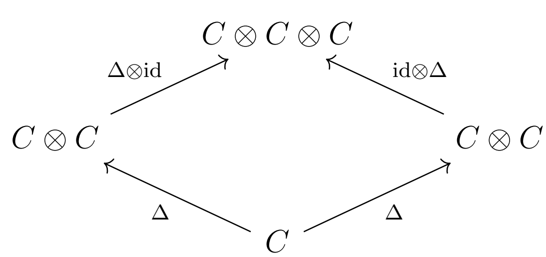
The unit is given by $\eta(k) = k$.
We would now like to state the universal property of tensor algebras. It will be important in the following.

Theorem 8.1 (Universal property of tensor algebras)**.** Let V be a vector space over k, and $T(V)$ be the tensor algebra of V. Then any linear transformation $f : V \to A$ from V to an algebra A over k can be uniquely extended to an algebra homomorphism $\tilde{f} : T(V) \to A$ indicated by the following commutative diagram:



This theorem will be particularly important when $A = T(V)$.

We would now like to define the dual notation of an algebra, namely a coalgebra. A coalgebra C over a field k is a k-vector space with a k-linear map $\Delta : C \to C \otimes C$ called the coproduct. All our coalgebras will be assumed to be coassociative and with counit. In other words, the diagram



commutes, and there exists a k-linear map $\epsilon : C \to k$, called the counit, such that the two diagrams below commute

$$\begin{array}{ccc} C \otimes C & \xleftarrow{\ \Delta\ } & \\ {\scriptstyle \epsilon\otimes \mathrm{id}}\downarrow & \nwarrow & \\ k \otimes C & \xrightarrow[\sim]{} & C \end{array} \qquad \begin{array}{ccc} C \otimes C & \xleftarrow{\ \Delta\ } & \\ {\scriptstyle \mathrm{id}\otimes \epsilon}\downarrow & \nwarrow & \\ C \otimes k & \xrightarrow[\sim]{} & C \end{array}$$

If we label the first copy in $C \otimes C$ by $C_{(1)}$ and the second copy by $C_{(2)}$ then we may describe the comultiplication map Δ as

$$\Delta : C \to C_{(1)} \otimes C_{(2)} \qquad c \mapsto \sum_{i=1}^{n} c^i_{(1)} \otimes c^i_{(2)}.$$ [36]

This is often too verbose in the literature. Comultiplication will usually be written as

$$\Delta(c) = c^i_{(1)} \otimes c^i_{(2)} \quad \text{or} \quad \Delta(c) = c_{(1)} \otimes c_{(2)}$$

where the latter notation is evil but indicates that the structure is completely linear. Either of the two latter notations are called "Sweedler Notation".[37]

A coalgebra morphism (or coalgebra homomorphism) $f : C \to C'$ is a k-linear map such that $\Delta' \circ f = (f \otimes f) \circ \Delta$ and $\epsilon' \circ f = \epsilon$ where Δ' and ϵ' are the comultiplication and counit of C' respectively.

We would like to note that given two algebras A_1 and A_2 (or coalgebras C_1 and C_2) we can construct a new algebra (or coalgebra) where the underlying vector space is $V_1 \otimes V_2$ (or $C_1 \otimes C_2$) with multiplication map defined by

$$m((a_1 \otimes a_2) \otimes (a_1' \otimes a_2')) \coloneqq m_1(a_1 \otimes a_1') \otimes m_2(a_2 \otimes a_2')$$

(or comultiplication defined by $\Delta(c_1 \otimes c_2) \coloneqq \Delta_1(c_1) \otimes \Delta_2(c_2)$) where m_1 and m_2 are the multiplications on A_1 and A_2 respectively (or Δ_1 and Δ_2 are the comultiplications on C_1 and C_2 respectively). The units (or counits) are obvious.

Example 8.2 (Coalgebras)**.** We will be primarily concerned with two examples of coalgebras.

- The algebra $k[G]$ can also be realized as a coalgebra with comultiplication defined on the basis by $\Delta(e_g) = e_g \otimes e_g$.[38] The counit is given by $\epsilon(e_g) = 1$.

[36] The superscript i is purely an index and is not denoting powers.
[37] We will often use Sweedler notation in the remainder of these notes for easy of notation.
[38] Elements x such that $\Delta(x) = x \otimes x$ are called grouplike.

- The tensor algebra $T(V)$ can also be realized as a coalgebra in two ways. For the first coalgebra, set

$$\Delta(v_1 \otimes \cdots \otimes v_\ell) = \sum_{j=0}^{\ell} (v_1 \otimes \cdots \otimes v_j) \otimes (v_{j+1} \otimes \cdots \otimes v_\ell),$$

where the empty tensors (occurring in the first and last terms of the sum) are 1. Then extend linearly. The counit is given by $\epsilon(m) = m$ for $m \in k$ and $\epsilon(v) = 0$ otherwise. This turns $T(V)$ into a coalgebra, but it turns out that with this structure $T(V)$ doesn't become any more interesting (as we will soon see).

The second coalgebra structure is given by first setting

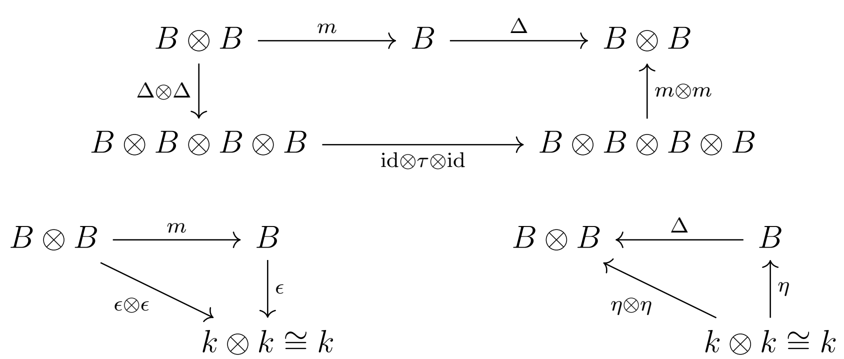
$$\Delta(v) = (v \otimes 1) + (1 \otimes v) \quad \text{and} \quad \Delta(1) = 1 \otimes 1$$

on $T^1(V)$ where we consider $v \otimes 1, 1 \otimes v, 1 \otimes 1 \in V \otimes V$.[39] This map is clearly k-linear and so extends a k-linear map $\Delta : T(V) \to T(V) \otimes T(V)$ by the universal property of tensor algebras. It can be checked that this map is given inductively by

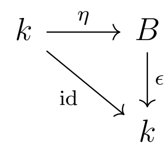
$$\Delta(v_1 \otimes v_2) = \Delta(v_1) \otimes \Delta(v_2)$$

for $v_1 \otimes v_2 \in T^2(V)$ and so on for $T^n(V)$. The counit is the same as before. This extension is the second coalgebra structure on $T(V)$. It will turn out to be the more useful one. If we have not made it clear enough: these two coalgebras are not the same even though they posses the same counit.

We also have the notion of a biaglebra. A bialgebra B over a field k is both an algebra over k and a coalgebra over k such that the following four diagrams commute:



[39] Observe $\Delta(v) \in V \otimes V$ by linearity of vector spaces. Also, elements x such that $\Delta(x) = (x \otimes 1) + (1 \otimes x)$ are called primitive.



In the first diagram

$$\tau : B \otimes B \to B \otimes B \qquad v \otimes w \mapsto w \otimes v$$

is the twist map. Observing these diagrams for a moment shows that they are natural compatibility conditions between the algebra and coalgebra structures of B. If these diagrams are commutative we often say that Δ and ϵ are algebra maps (or equivalently m and η are coalgebra maps).

A bialgebra morphism (or bialgebra homomorphism) $f : B \to B'$ is a k-linear map that is both an algebra morphism and a coalgebra morphism.

Example 8.3 (Bialgebras)**.** Below are our two primarily examples of bialgebras.

- The group algebra $k[G]$ can be considered as a bialgebra by directly checking the commutativity of the diagrams above.
- The second coalgebra structure on $T(V)$ makes it into a bialgebra while the first one does not. This is why the second structure, while more difficult to define, is more useful.

9 Hopf Algebras

We are ready to introduce Hopf algebras. They are bialgebras with additional structure. In particular, there exists a map $S : H \to H$ called the antipode such that

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \epsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta.$$

Equivalently, the following diagram commutes:

$$\begin{array}{ccccc} H & \xrightarrow{\epsilon} & k & \xrightarrow{\eta} & H \\ \Delta \downarrow & & & & \uparrow m \\ H \otimes H & & \xrightarrow[S\otimes \mathrm{id}]{\mathrm{id}\otimes S} & & H \otimes H \end{array}$$

A Hopf algebra morphism (or Hopf algebra homomorphism) $f : H \to H'$ is a k-linear map that is both an algebra morphism and a coalgebra morphism and satisfies $f \circ S = S' \circ f$ where S' is the antipode of H'. It so happens that the last property is automatically satisfied if f is a bialgebra morphism. Therefore, a Hopf algebra morphism is simply a bialgebra morphism.

At this point it is not out of the question to wonder why we would want such a map S. The idea is that S is a sort of “generalized inversion” for elements of H. In fact, it is a precise inverse if we interpret the commutative diagram above in another manner. Indeed, if H is a Hopf algebra then we can make $\mathrm{End}(H)$ into an associative unital algebra where the multiplication is defined by

$$m(f \otimes g) := m \circ (f \otimes g) \circ \Delta.$$

We often write $f * g$ for $m(f \otimes g)$ since the multiplication is a sort of convolution. Observe that $\eta \circ \epsilon$ is the identity in this algebra. Commutativity of the above diagram then reads

$$S * \mathrm{id} = \eta \circ \epsilon = \mathrm{id} * S.$$

In other words, S is the inverse of id in $\mathrm{End}(H)$. In some cases, S is the inverse on certain elements of H as we will see in the examples of Hopf algebras below:

Example 9.1. In our running example we will have the following Hopf algebras:

- We can make the bialgebra $k[G]$ into a Hopf algebra by declaring $S(e_g) = e_{g^{-1}}$ on the basis and then extending linearly.[40]

[40] The antipode map S is literal inversion here so $S^2 = \mathrm{id}$. This need not be true in general.

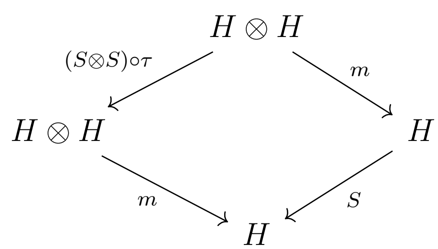
- We can make the bialgebra $T(V)$ into a Hopf algebra by first setting $S(v) = -v$ on $T(V)$. Since this map is k-linear, by the universal property of tensor algebras it extends to a unique algebra homomorphism $S : T(V) \to T(V)$. The antipode is declared to be this extension.

The antipode map S also has the following properties:

1. S is an antialgebra map. This means

$$S \circ m = m \circ (S \otimes S) \circ \tau,$$

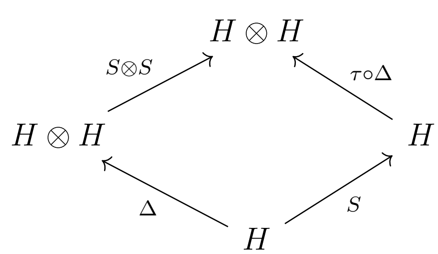
and $S \circ \eta = \eta$. The first identity can be expressed by the commutative diagram



2. S is an anticoalgebra map. This means

$$(S \otimes S) \circ \Delta = \tau \circ \Delta \circ S,$$

and $\epsilon \circ S = \epsilon$. The first identity can be expressed by the commutative diagram

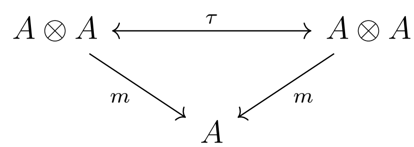


3. S is unique if it exists.[41]

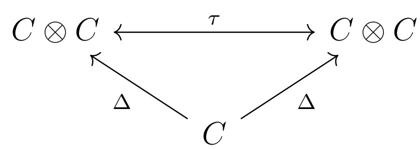
[41] This means there is at most one Hopf algebra structure on any bialgebra. In other words, being a Hopf algebra is a property of being a bialgebra.

We would like to state two more facts about Hopf algebras. The first will provide a sufficient condition for S to be an involution, and the second will tell us about how certain left H-modules. To do so we first need to introduce a property of algebras, coalgebras, and bialgebras. Let us discuss algebras and coalgebras first.

We say an algebra A is commutative if the algebra multiplication is commutative. In other words, we have the commutative diagram



Similarly, we say a coalgebra C is cocommutative if the comultiplication is commutative. In other words, we have the following diagram commutes:



We say that a bialgebra or Hopf algebra is commutative or cocommutative if its algebra or coalgebra structure is commutative or cocommutative respectively.

Below is an example of these properties:

Example 9.2.

- If G is abelian, then $k[G]$ is commutative. Also, $k[G]$ is always cocommutative.
- $T(V)$ is commutative if and only if $\dim(V) \leq 1$. Also, $T(V)$ is always cocommutative.

The property we want to discuss for bialgebras is more subtle. Given a bialgebra B and two left B-modules V and W, $V \otimes W$ has a natural left $(B \otimes B)$-module structure. By composing with comultiplication, we get a natural B-algebra structure on $V \otimes W$ defined by

$$b(v \otimes w) \coloneqq \Delta(b)(v \otimes w)$$

where $\Delta(b)(v \otimes w)$ acts by the natural left $(B \otimes B)$-module structure on $V \otimes W$.

We can now state the properties of Hopf algebras discussed earlier (one of which answers the previous question in the setting of Hopf algebras):

1. If H is commutative or cocommutative then $S^2 = \text{id}$.

2. If H is cocommutative, then then H-structure on $V \otimes W$ (where V and W are left H-modules) is not isomorphic to that of $W \otimes V$.

For quantum group we would like to relax the condition that $V \otimes W \cong W \otimes V$ as left H-modules (i.e., relax cocommutativity).

A Lie algebra $\mathfrak{g}$ is a vector space with a bracket operation $[\cdot,\cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$. We can form the tensor algebra $T(\mathfrak{g})$ and quotient by the ideal generated by all relations of the form $[g_1, g_2] - (g_1 \otimes g_2) + (g_2 \otimes g_1)$ to get the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$[42] as explained previously. This is an interesting algebra because of the following comments:

1. Any theorem which is true for both group algebras and universal enveloping algebras is true for all cocommutative Hopf algebras.

2. Sweedler has a book with many results on the classification of finite dimensional Hopf algebras, but there remain many open questions.

3. There was very little supply of noncommutative and noncocommutative Hopf algebras until Drinfeld and Jimbo gave examples examples of $\mathcal{U}_q(\mathfrak{g})$ for nice Lie algebras $\mathfrak{g}$.

[42]This algebra contains all the of representation theory of $\mathfrak{g}$.

10 Left Modules and Dual Hopf Algebras

We discuss left modules first. Given to left H-modules V and W, $V \otimes W$ inherits a left H-module structure. If H is cocommutative $V \otimes W \cong W \otimes V$ as left H-modules. In general, $V \otimes W$ and $W \otimes V$ may be quite different as left H-modules.[43]

If M is a left H-module, consider the dual $M^* = \mathrm{Hom}(M, k)$. We can define a left action on M^* given by

$$(hf)(m) \coloneqq f(S(h)m).$$

If S is invertible, then we can define a second left action by

$$(hf)(m) \coloneqq f(S^{-1}(h)m).$$

These two actions do not give rise to isomorphic H-modules in general. This action will be particularly important when we consider H as a left H-module

Now we discuss dual Hopf algebras. In the setting of finite dimensional Hopf algebras (which we are rarely in) we have a dual space $H^* = \mathrm{Hom}(H, k)$ and a canonical isomorphism between $(H^*)^*$ and H. We also have a natural map

$$\langle \cdot, \cdot \rangle : H^* \otimes H \to k \qquad \langle \phi, v \rangle \mapsto \phi(v)$$

called the evaluation map. We can define a Hopf algebra structure on H^* in terms of the evaluation map by making the following definitions

$$\begin{aligned}
(m^*(\phi \otimes \psi))(h) &\coloneqq \langle \phi \otimes \psi, \Delta(h) \rangle, \\
(\eta^*(a))(h) &\coloneqq \langle a\epsilon, h \rangle, \\
(\Delta^*(\phi))(h \otimes g) &\coloneqq \langle \phi, m(h \otimes g) \rangle, \\
\epsilon^*(\phi) &\coloneqq \langle \phi, 1 \rangle, \\
(S^*(\phi))(h) &\coloneqq \langle \phi, S(h) \rangle.
\end{aligned}$$

In the infinite dimensional setting, we only know $(H \otimes H)^* \cong H^* \otimes H^*$. So, we say that two Hopf algebras H and H' are dually paired if there exists a nondegenerate pairing $\langle \cdot, \cdot \rangle : H \otimes H \to k$[44] satisfying the five equations above (viewed as identities and not equations). In general, H may have several dual pairings.

We would like to give several examples of dual pairings.

Example 10.1 (Dual pairings)**.**

[43] When we define quantum groups, we want to limit how bad this difference can be.
[44] This is not necessarily an evaluation map.

- As we have already said, if H is a finite dimensional Hopf algebra it has a dual Hopf algebra H^* which is automatically dually paired to H.

- Consider the Hopf algebra $k[G]$. It is dually paired with the algebra of functions on G, denoted $k(G)$, which can be made into a Hopf algebra.

- If $\mathfrak{g}$ is a finite dimensional complex semisimple Lie algebra (think $\mathfrak{sl}(2, \mathbb{C})$) with Lie group G, then the dual of $\mathcal{U}(\mathfrak{g})$ is the coordinate algebra $\mathbb{C}[G]$ for G over $\mathbb{C}$ defined by
$$\mathbb{C}[G] := \mathbb{C}[x_{i,j}]_{1 \leq i \leq j \leq n} / (p(\underline{x}))$$
where $\underline{x} = (x_{i,j})_{1 \leq i \leq j \leq n}$ and $(p(\underline{x}))$ is the ideal generated by the polynomial equations which give embeddings of G into $\mathrm{Mat}(n, \mathbb{C})$. Its coalgebra structure is given on generators by
$$\Delta(x_{i,j}) = \sum_{k=1}^{n} (x_{i,k} \otimes x_{k,j}) \quad \text{and} \quad \epsilon(x_{i,j}) = \delta_{i,j}$$
where $\delta_{i,j}$ is the Kronecker delta. We omit discussing the antipode S because it is more complicated to define and dependent on cofactors of the matrix $[x_{i,j}]$. The pairing between these Hopf algebras is given by extending the pairing
$$\langle \alpha, x_{i,j} \rangle = \rho(\alpha)_{i,j},$$
where $\rho : \mathfrak{g} \to \mathrm{Mat}(n, \mathbb{C})$ is the defining representation, to $\mathcal{U}(\mathfrak{g})$.

11 The Hopf Algebra $\mathcal{U}_q(\mathfrak{b}_+)$

We will discuss a new Hopf algebra over $\mathbb{C}$. For reasons which will become clear, define $\mathcal{U}_q(\mathfrak{b}_+)$ by

$$\mathcal{U}_q(\mathfrak{b}_+) := \langle X, K, K^{-1} \mid KK^{-1} = 1 = K^{-1}K, KX = qXK\rangle^{45},$$

and define maps Δ, ϵ, and S by

$$\Delta(X) := (X \otimes 1) + (K \otimes X), \quad \Delta(K) := K \otimes K, \quad \Delta(K^{-1}) := K^{-1} \otimes K^{-1},$$
$$\epsilon(X) := 0, \quad \epsilon(K) := \epsilon(K^{-1}) = 1,$$
$$S(X) := -K^{-1}X, \quad S(K) := K^{-1}, \quad S(K^{-1}) := K.$$

It is a fact that this set of relations and definitions extends to define an infinite dimensional noncommutative noncocommutative Hopf algebra (the multiplication and unit are the obvious ones). We sketch a proof of this fact below.

Theorem 11.1. The definitions of Δ, ϵ, and S for $\mathcal{U}_q(\mathfrak{b}_+)$ on X, K, and K^{-1} extend to make $\mathcal{U}_q(\mathfrak{b}_+)$ a Hopf algebra.

Proof sketch. Extend Δ and ϵ and check that this gives algebra maps. Then extend S as an antialgebra map and check the axioms for S on generators. Then the axioms for S on products will follow automatically because Δ and ϵ are well-defined multiplicatively. The facts that $\mathcal{U}_q(\mathfrak{b}_+)$ is infinite dimensional, noncommutative, and noncocommutative can all be directly verified. □

It can also be checked that for any $u \in \mathcal{U}_q(\mathfrak{b}_+)$ we have $S^2(u) = K^{-1}uK$ so S is bijective and hence invertible.

It is best to think of $\mathcal{U}_q(\mathfrak{b}_+)$ as a q-deformation of the universal enveloping algebra $\mathcal{U}(\mathfrak{b}_+)$ where $\mathfrak{b}_+$ is the Lie algebra of the Boreal subgroup of $\mathrm{SL}(2,\mathbb{C})$. For the reader unfamiliar with Borel subgroups, the Boreal subgroup B of $\mathrm{SL}(2,\mathbb{C})$ is

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a, b \in \mathbb{C} \right\}.$$

The Lie algebra $\mathfrak{b}_+$ has as generators

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

[45] Here q is an arbitrary nonzero element of k.

In the deformation $D \to K, K^{-1}$ and $X \to X$. It is a general fact that $\mathcal{U}_q(\mathfrak{b}_+)$ is self-dual. We would like to explore this fact, but first let us provide reasoning for why duality is important.

Hopf duality sets up a duality between left H-modules and right H-comodules.[46] Moreover, there are several constructions of $\mathcal{U}_q(\mathfrak{g})$ using duality. We state two below:

- There is a natural action of the Hopf algebra $\mathrm{SL}_q(2)$[47] on the quantum plane[48], and the dual is $\mathcal{U}_q(\mathfrak{sl}(2,k))$.
- There is a construction of Drinfeld where given a Hopf algebra H and a dual H^* of H, one can form a new Hopf algebra $D(H)$. For finite dimensional H and H^*, $D(H)$ is guaranteed to be a quasi-triangular Hopf algebra.[49]

Let's being our discussion of $\mathcal{U}_q(\mathfrak{b}_+)$ with a couple of facts.

Theorem 11.2. $\mathcal{U}_q(\mathfrak{b}_+)$ has $\{X^m K^n\}_{\substack{m\geq 0\\ n\in\mathbb{Z}}}$ as a basis.

Proof sketch. Show $\{X^m K^n\}_{\substack{m\geq 0\\ n\in\mathbb{Z}}}$ is a spanning set by checking that monomials are stable under multiplication by any element of $\mathcal{U}_q(\mathfrak{b}_+)$ (we can check this on generators). For linear independence, set $R = k[A, B, B^{-1}]$. Observe R is a commutative ring with basis $\{A^m B^n\}_{\substack{m\geq 0\\ n\in\mathbb{Z}}}$. We have endomorphisms $f, g : R \to R$ defined on basis elements by

$$f(A^m B^n) := A^{m+1}B^n \quad \text{and} \quad g(A^m B^n) := q^m A^m B^{n+1}.$$

Observe g has an inverse g^{-1} defined by

$$g^{-1}(A^m B^n) := q^{-m} A^m B^{n-1}.$$

Check $g \circ f = q(f \circ g)$. This implies that we have a map $\mathcal{U}_q(\mathfrak{b}_+) \to \mathrm{End}_k(R)$ defined on generators by $X \mapsto f$, $K \mapsto g$, and $K^{-1} \mapsto g^{-1}$. Now check f, g, and g^{-1} are linearly independent in $\mathrm{End}_k(R)$ implying X, K, and K^{-1} are linearly independent in $\mathcal{U}_q(\mathfrak{b}_+)$. □

The second fact is as follows:

[46]This is to be defined.

[47]We have not discussed this Hopf algebra yet.

[48]Think of the affine plane with the relation $xy = qyx$.

[49]We have not yet defined what it means for a Hopf algebra to be quasi-triangular.

Proposition 11.1. In $\mathcal{U}_q(\mathfrak{b}_+)$ we have the identity

$$\Delta(X^m) = \sum_{r=0}^{m} \begin{bmatrix} r \\ m \end{bmatrix}_q X^{m-r}(K^r \otimes X^r),$$

where we make definitions

$$\begin{bmatrix} r \\ m \end{bmatrix}_q := \frac{[m]_q!}{[r]_q![m-r]_q!},$$

$[r]_q! := [r]_q[r]_{q-1} \cdots [1]_q$, and $[r]_q := (1-q^r)/(1-q)$.[50] We also make the convention that

$$\begin{bmatrix} m \\ m \end{bmatrix}_q := 1 =: \begin{bmatrix} 0 \\ m \end{bmatrix}_q.$$

Proof sketch. Recall that Δ is given by extending it on basis elements multiplicatively with the base condition $\Delta(X) = (X \otimes 1) + (K \otimes X)$. Then apply the q-binomial formula to summands and use induction: if $qAB = BA$, then

$$(A+B)^n = \sum_{m=0}^{n} \left[\frac{n}{m}\right]_q A^m B^{n-m}.$$

□

With these two properties we can show $\mathcal{U}_q(\mathfrak{b}_+)$ is self-dual. Indeed, we have the following proposition and proof sketch due to Majid:

Proposition 11.2. The identities

$$\langle K, K \rangle = q, \quad \langle X, X \rangle = 1, \quad \text{and} \quad \langle X, K \rangle = \langle X, K \rangle = 1,$$

uniquely determines non-degenerate pairing between $\mathcal{U}_q(\mathfrak{b}_+)$ and itself satisfying the properties of a dual pairing.

Proof sketch. We're going to construct an extension of the pairing by assuming some functions satisfy the pairing properties, define the pairing using this extension, and then recheck that it satisfies all the properties. For $m \geq 0$ and $n \in \mathbb{Z}$, define functions

$$f_{m,n}(u) := \langle X^m K^n, u \rangle,$$

[50] We say $\begin{bmatrix} r \\ m \end{bmatrix}_q$ is a q-binomial coefficient.

and assume they satisfy the pairing properties as well as the identities in the proposition. First evaluate $f_{m,n}$ on X and K. It can be checked that

$$f_{m,n}(K) = q^n \delta_{m,0} \quad \text{and} \quad f_{m,n}(X) = \delta_{m,1}$$

where δ is the Kronecker delta. Now show for any $u, u' \in \mathcal{U}_q(\mathfrak{b}_+)$, that

$$f_{m,n}(uu') = \sum_{r=0}^{m} \begin{bmatrix} r \\ m \end{bmatrix}_q f_{m-r,n+r}(u) f_{r,n}(u').$$

Then extend the pairing by the identity above (which uniquely determines it). Now go back and check all the pairing properties are satisfied on the basis elements $\{X^m K^n\}_{\substack{m \geq 0 \\ n \in \mathbb{Z}}}$. □

12 Braidings and Quasitriangular Hopf Algebras

We are going to define quantum groups in this section, but to do that we first need to introduce quasitriangular Hopf algebras.

Recall that if B is a bialgebra, then given two left B-algebras V and W we can form a left B-algebra $V \otimes W$. In category-theoretic language we say that left B-modules with left B-module homomorphisms form a left monodial[51] category. Since comultiplication is associative, the operation $(V, W) \mapsto V \otimes W$ is associative. So we really have a associative left monodial category. Also recall that $V \otimes W$ and $W \otimes V$ may have very different structures as left B-modules. Moreover, if B is cocommutative then $\tau : V \otimes W \to W \otimes V$ gives a left B-module isomorphism between $V \otimes W$ and $W \otimes V$. Now suppose that $V \otimes W \cong W \otimes V$ via some (not necessarily τ) left B-module isomorphism. More generally, suppose we have a natural family of left B-module isomorphisms $t_{V,W} : V \otimes W \to W \otimes V$, for all B-modules V and W, that is compatible with associativity.[52] By this we mean that the following diagrams are commutative

$$
\begin{array}{ccccccc}
 & & V \otimes (W \otimes X) & \xrightarrow{t_{V,W\otimes X}} & (W \otimes X) \otimes V & & \\
 & \nearrow & & & & \searrow & \\
(V \otimes W) \otimes X & & & & & & W \otimes (X \otimes V) \\
 & \searrow^{t_{V,W}\otimes \mathrm{id}} & & & & \nearrow^{\mathrm{id}\otimes t_{V,X}} & \\
 & & (W \otimes V) \otimes X & \longrightarrow & W \otimes (V \otimes X) & &
\end{array}
$$

$$
\begin{array}{ccccccc}
 & & V \otimes (W \otimes X) & \xleftarrow{t_{W\otimes X,V}} & (W \otimes X) \otimes V & & \\
 & \swarrow & & & & \nwarrow & \\
(V \otimes W) \otimes X & & & & & & W \otimes (X \otimes V) \\
 & \nwarrow^{t_{W,V}\otimes \mathrm{id}} & & & & \swarrow^{\mathrm{id}\otimes t_{X,V}} & \\
 & & (W \otimes V) \otimes X & \longleftarrow & W \otimes (V \otimes X) & &
\end{array}
$$

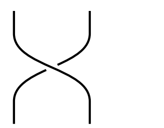
where all the unlabeled arrows are the obvious associative isomorphisms on tensor products of modules. This more general structure, in the category theoretic language,

[51]We say monodial since the operation taking V and W to $V \otimes W$ defines a monoid on the space of B-modules.

[52]This is more general since we can think of $\tau : V \otimes W \to W \otimes V$ as a family of isomorphisms if we vary V and W across all left B-modules.

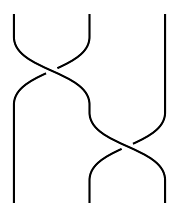
is called a braided left monodial category. We give an example to illustrate why we call this category braided.

Example 12.1 (Braided monodial category)**.** The prototypical example of a braided monodial category comes from braids on n-strings which justifies the name. Let the objects of this category be elements of $\mathbb{Z}_{\geq 0}$ and the morphisms be braids between them.[53] For example, the braid

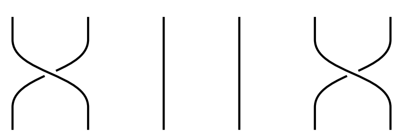


is an example of a morphism between the object 3. Composition of braids is given by stacking. For example composing the braids

produces the braid



The monodial operation is given by concatenating strings. For example, concatenation of the two braids above (instead of composition) produces the braid



We encourage the interested reader to deduce what the isomorphism t_{b_1,b_2} is for arbitrary braids b_1 and b_2.

[53]By the definition of a braid this means there are only morphisms between m and n if $m = n$.

If $t_{V,W} : V \otimes W \to W \otimes V$ is a braiding[54], consider $t_{B,B} : B \otimes B \to B \otimes B$ and let $R = t_{B,B}(1 \otimes 1)$. Conversely given any $R = \sum_{i=1}^{n} b^i_{(1)} \otimes b^i_{(2)} \in B \otimes B$, we can construct a family of maps

$$t^{(R)}_{V,W} : V \otimes W \to W \otimes V \qquad v \otimes w \mapsto \sum_{i=1}^{n}((b^i_{(1)}w) \otimes (b^i_{(2)}v)).$$

In fact, this family of maps is a braiding if and only if R satisfies the following three properties

1. R is invertible.
2. $\tau \circ \Delta = R\Delta R^{-1}$, where $R\Delta R^{-1}$ takes b to $R\Delta(b)R^{-1}$ for all $b \in B$.
3. $(\Delta \otimes \mathrm{id})(R) = R_{(1,3)}R_{(2,3)}$ and $(\mathrm{id} \otimes \Delta)(R) = R_{(1,3)}R_{(1,2)}$ where $R_{(i,j)}$ is the image of R under the map $\phi_{(i,j)} : B \otimes B \to B \otimes B \otimes B$ which acts by the identity on the i and j copies of B inside $B \otimes B \otimes B$ and trivially on the other copy.

Moreover, there is a bijective correspondence between R satisfying these properties and braidings.

We may now define a quasitriangular Hopf algebra over k. A quasitriangular Hopf algebra over k (also known as a quantum group over k) is a pair (H, R) consisting of a Hopf algebra H over k and an element $R \in H \otimes H$[55] such that the three properties above are satisfied.[56] At this point, we encourage the reader to refer to the appendix if they wish to learn why we call a quasitriangular Hopf algebra a quantum group.

[54]By this we mean the family of maps $t_{V,W} : V \otimes W \to W \otimes V$ gives rise to a braided left monodial category of left B-modules and left B-module homomorphisms.

[55]This R is sometimes called the universal R-element.

[56]We will not go into detail about why we call this structure quasitriangular.

13 Quantum Groups and Quantum Yang-Baxter Equations

We would like to state and sketch the proof of a lemma about quantum groups.

Lemma 13.1. If (H, R) is a quantum group, then we have the following:

1. $(\epsilon \otimes \mathrm{id})(R) = (\mathrm{id} \otimes \epsilon)(R) = 1$, $(S \otimes \mathrm{id})(R) = R^{-1}$, and $(\mathrm{id} \otimes S)(R^{-1}) = R$.
2. $(H, \tau(R^{-1}))$ is a quantum group.
3. In $H \otimes H \otimes H$ we have the abstract quantum Yang-Baxter equation
$$R_{(1,2)}R_{(1,3)}R_{(2,3)} = R_{(2,3)}R_{(1,3)}R_{(1,2)}.$$

Proof sketch. We will sketch each property individually.

1. We will show $(e \otimes \mathrm{id})(R) = 1$; the other cases are handled similarly. By the properties of quantum groups we have
$$(\Delta \otimes \mathrm{id})(R) = R_{(1,3)}R_{(2,3)} \quad \text{and} \quad (\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id}.$$
Apply $(\epsilon \otimes \mathrm{id}) \circ \mathrm{id}$ to both side of the first of the two identities above. The right-hand side is R. The left-hand side is
$$((\epsilon \otimes \mathrm{id}) \circ \mathrm{id})(R_{(1,3)}R_{(1,2)}) = (\epsilon \otimes \mathrm{id})(R)\epsilon(1)R.$$
Now use the fact that R is invertible.
2. This is straightforward to check.
3. We know
$$R_{(1,2)}R_{(1,3)}R_{(2,3)} = R_{(1,2)}(\Delta \otimes \mathrm{id})(R) \quad \text{and} \quad \tau \circ \Delta = R_{(1,2)}\Delta R_{(1,2)}^{-1}$$
where $R_{(1,2)}\Delta R_{(1,2)}^{-1}$ sends h to $R_{(1,2)}\Delta(h)R_{(1,2)}^{-1}$ for all $h \in H$. Therefore
$$\begin{aligned} R_{(1,2)}R_{(1,3)}R_{(2,3)} &= R_{(1,2)}(\Delta \otimes \mathrm{id})(R) \\ &= (\tau \otimes \Delta \otimes \mathrm{id})(R)R_{(1,2)} \\ &= ((\tau \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id}))(R)R_{(1,2)} \\ &= (\tau \otimes \mathrm{id})(R_{(1,3)}R_{(2,3)})R_{(1,2)} \\ &= R_{(2,3)}R_{(1,3)}R_{(1,2)}. \end{aligned}$$
In the second to last line τ flipping the subscripts of the $R_{(i,j)}$.

□

Observe that this connects quantum groups and quantum Yang-Baxter equations! We have now shown that a quantum group gives rise to an abstract quantum Yang-Baxter equation. In fact, we get an honest quantum Yang-Baxter equation by choosing a representation (ρ, V) for H and then applying $\rho \otimes \rho$ to R. That is, the matrix $(\rho \otimes \rho)(R)$ satisfies a quantum Yang-Baxter equation.

14 An Investigation of $\mathcal{U}_q(\mathfrak{sl}(2,k))$

While we have defined quantum groups and stated some of their properties we have not yet given any examples. In the following we would like to do an in-depth investigation.

Recall the algebra

$$\mathcal{U}_q(\mathfrak{sl}(2,k)) = \langle E, F, K, K^{-1} \mid {}_{KFK^{-1}=q^{-2}F,[E,F]=EF-FE=(K-K^{-1})/(q-q^{-1})}^{KK^{-1}=1,K^{-1}K=1,KEK^{-1}=q^2E,} \rangle.$$

This construction makes sense over any characteristic 0 field k with $q \neq 0, 1, -1$ (for example take $k = \mathbb{C}$ and $q \in \mathbb{C}^*$ not a unit). We would like to answer the following questions about $\mathcal{U}_q(\mathfrak{sl}(2,k))$:

1. Does $\mathcal{U}_q(\mathfrak{sl}(2,k))$ have a Hopf algebra structure?
2. Is $\mathcal{U}_q(\mathfrak{sl}(2,k))$ really a q-deformation of $\mathcal{U}(\mathfrak{sl}(2,k))$?
3. What are the finite dimensional left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-modules and how do they compare to the left $\mathcal{U}(\mathfrak{sl}(2,k))$-modules?[57]
4. Is $\mathcal{U}_q(\mathfrak{sl}(2,k))$ quantum group and can we determine a method to compute R?
5. Can we construct $\mathcal{U}_q(\mathfrak{sl}(2,k))$ by some natural process?

We will answer these questions over the next few sections.

To answer the first, we do get a Hopf algebra structure. The idea here mimics that of defining a Hopf algebra structure on $\mathcal{U}_q(\mathfrak{b}_+)$. We define Δ and ϵ on generators and extended multiplicatively, define s on generators and extend as an antialgebra, then check all relations. In particular, we define

$$\begin{gathered}\Delta(E) \coloneqq (E \otimes 1) + (K \otimes E), \quad \Delta(F) \coloneqq (F \otimes K^{-1}) + (1 \otimes F), \\ \Delta(K) \coloneqq K \otimes K, \quad \Delta(K^{-1}) \coloneqq K^{-1} \otimes K^{-1}, \\ \epsilon(E) \coloneqq \epsilon(F) = 0, \quad \epsilon(K) \coloneqq \epsilon(K^{-1}) \coloneqq 1, \\ S(E) \coloneqq -K^{-1}E, \quad S(F) \coloneqq -FK, \quad S(K) \coloneqq K^{-1}, \quad S(K^{-1}) \coloneqq K.\end{gathered}$$

The answer to this question is "sort of". We can present $\mathcal{U}(\mathfrak{sl}(2,k))$ as

$$\mathcal{U}(\mathfrak{sl}(2,k)) = \langle X, Y, H \mid [X,Y] = H, [H,X] = 2X, [H,Y] = -2Y \rangle.^{58}$$

[57] This will come down to two cases, either q is a root of unity or it is not.

[58] Here X is similar to E, Y is similar to F, and H is similar to K and K^{-1} in $\mathcal{U}_q(\mathfrak{sl}(2,k))$.

We would hope that $\mathcal{U}_q(\mathfrak{sl}(2,k)) \to \mathcal{U}(\mathfrak{sl}(2,k))$ as $q \to 1$, but this is broken because $q - q^{-1}$ is in the denominator of a relation for our presentation of $\mathcal{U}_q(\mathfrak{sl}(2,k))$. Here's how we fix this issue. It can be shown that $\mathcal{U}_q(\mathfrak{sl}(2,k))$ is isomorphic (as an algebra) to another algebra $\mathcal{U}'_q$ defined by

$$\mathcal{U}'_q \coloneqq \langle E, F, K, K^{-1}, L \mid {}^{R(\mathcal{U}_q(\mathfrak{sl}(2,k))), [E,F]=L, (q-q^{-1})L = K-K^{-1},}_{[L,E]=q(EK+K^{-1}E), [L,F]=-q^{-1}(FK+K^{-1}F)} \rangle$$

where $R(\mathcal{U}_q(\mathfrak{sl}(2,k)))$ is representing all the relations for $\mathcal{U}_q(\mathfrak{sl}(2,k))$.[59] The isomorphism $\varphi : \mathcal{U}' \to \mathcal{U}_q(\mathfrak{sl}(2,k))$ is defined by $E \mapsto E$, $F \mapsto F$, $K \mapsto K$, $K^{-1} \mapsto K^{-1}$, and $L \mapsto [E,F]$. Now $\mathcal{U}'_q$ at $q = 1$ satisfies $\mathcal{U}'_1 \cong \mathcal{U}(\mathfrak{sl}(2,k))[K]/(K^2-1)$ so we get a projection onto $\mathcal{U}_q(\mathfrak{sl}(2,k))$ via a map defined by $E \mapsto X$, $F \mapsto Y$, $K \mapsto 1$, and $L \mapsto H$.

[59] We don't use this algebra because there are far more relations to check.

15 The Representation Theory of $\mathcal{U}(\mathfrak{sl}(2,k))$

We'd like to understand the representation theory of $\mathcal{U}(\mathfrak{sl}(2,k))$. We will need the following preliminary result before we dig into the representation theory:

Lemma 15.1. We have the two identities:

$$X^p H^q = (H-2p)^q X^p \quad \text{and} \quad Y^p H^q = (H+2p)^q Y^p.$$

Proof sketch. Use induction. □

A basis theorem will also be needed:

Theorem 15.1. $\{X^i Y^j H^k\}_{i,j,k\in\mathbb{Z}_{\geq 0}}$ is a basis for $\mathcal{U}(\mathfrak{sl}(2,k))$.

Proof sketch. This is in the same spirit as proving a basis for $\mathcal{U}_q(\mathfrak{b}_+)$ done earlier. □

Finally, we will need a result about the center of $\mathcal{U}(\mathfrak{sl}(2,k))$:

Theorem 15.2. $C \coloneqq XY + YX + \frac{H^2}{2}$ is in the center of $\mathcal{U}(\mathfrak{sl}(2,k))$.[60] In fact, C generates the center.

Proof sketch. All that needs to be checked is that brackets with C vanish, and checking this on generators is enough. The second statement follows from a theorem of Harish and Chandra. □

We would like to use these results to determine all the finite dimensional left $\mathcal{U}(\mathfrak{sl}(2,k))$-modules (throughout we assume $k = \mathbb{C}$ for simplicity). The theory of highest weight vectors and highest weight modules will be useful here. We recast the main definitions in terms of $\mathcal{U}(\mathfrak{sl}(2,k))$ for brevity. If V is a finite dimensional left $\mathcal{U}(\mathfrak{sl}(2,k))$-module, a nonzero vectors $v \in V$ is said to be of weight $\lambda \in \mathbb{C}$ if $Hv = \lambda v$. It is called a highest weight vector if in addition $Xv = 0$. We say V is a highest weight representation of $\mathcal{U}(\mathfrak{sl}(2,k))$ if it is generated by a highest weight vector.

We now state and sketch the proof of a proposition:

Proposition 15.1. Every finite dimensional left $\mathcal{U}(\mathfrak{sl}(2,k))$-module V has a highest weight vector.

[60] C is called the Casimir element.

Proof sketch. H has some eigenvector w with an eigenvalue say α.[61] If $xw = 0$ we are done. If not, consider the sequence $\{X^n w\}_{n\geq 0}$. By the first result, $X^n w$ is an eigenvector of H with eigenvalue $\alpha + 2n$. But v is finite dimensional so there can only be finitely many distinct eigenvalues so there exists an n such that $X^n w \neq 0$ and $X^m w = 0$ for $m > n$. This implies $X^{n+1}w$ is a highest weight vector. □

We can now state the main structure theorem for finite dimensional left $\mathcal{U}(\mathfrak{sl}(2,k))$-modules.

Theorem 15.3 (Structure Theorem for finite dimensional left $\mathcal{U}(\mathfrak{sl}(2,k))$-modules)**.** The finite dimensional simple left $\mathcal{U}(\mathfrak{sl}(2,k))$-modules (up to isomorphism) are indexed by nonnegative integers n, call them $V(n)$, of dimension $n+1$ with highest weight vector v_n of weight n, and with the weight vectors of weights $n, n-2, \dots, -n$ forming a $\mathbb{C}$-basis for $V(n)$. In particular, the action of X raises the weight of a weight vector by 2, the action of Y on a weight vector lowers the weight by 2, and C acts on the $V(n)$ by scalars where the scalar is $n(n+2)/2$.[62]

We note that the actions of X and Y do not undo one another. For example, if V is a finite dimensional simple left $\mathcal{U}(\mathfrak{sl}(2,k))$-module, with highest weight vector v, then $V = \langle v \rangle = \langle XYv \rangle$, but $XYv = v$ need not occur. However, the weight spaces are 1-dimensional so XYv and v differ by a scalar. Moreover, with this theorem $\{v_p = Y^p v_n / p!\}_{0 \leq p \leq n}$ is a $\mathbb{C}$-basis for $V(n)$.

It is natural to ask the following question: given two finite dimensional simple left $\mathcal{U}(\mathfrak{sl}(2,k))$-modules $V(n)$ and $V(m)$, what is the decomposition of $V(n) \otimes V(m)$? Luckily, in this setting the question is not so difficult to answer as we have the following theorem:

Theorem 15.4 (Clebsch-Gordon Formula)**.** Given two finite dimensional simple left $\mathcal{U}(\mathfrak{sl}(2,k))$-modules $V(n)$ and $V(m)$,

$$V(n) \otimes V(m) \cong V(n+m) \otimes V(n+m-2) \otimes \cdots \otimes V(n-m).$$

Proof sketch. Check that the dimension of both sides is $(n+1)(m+1)$. This means we are done if we can find highest weight vectors for weights $n+m-2p$ where $0 \leq p \leq m$ in $V(n) \otimes V(m)$. If v and v' are the highest weight vectors for $V(n)$ and

[61] This relies on the fact that $k = \mathbb{C}$ is algebraically closed. The proof does hold in the general setting however.

[62] This last fact may be used to prove that any finite dimensional left $\mathcal{U}(\mathfrak{sl}(2,k))$-module is semisimple.

$V(m)$ respectively, let v_p and v'_p be the corresponding $\mathbb{C}$-basis vectors. Check that

$$\sum_{i=0}^{p} \frac{(m-p+i)!(n-i)!}{(m-p)!n!}(v_i \otimes v'_{p-i})$$

for $0 \leq p \leq m$ are the desired highest weight vectors. □

In general, quantum groups (like $\mathcal{U}_q(\mathfrak{sl}(2,k))$) will give rise to beautiful algorithms for computing decompositions of tensor products using bases like those just discussed.[63]

We end with a definition introducing the Hopf algebra point of view. If H is a Hopf algebra and A is an algebra (both over k) then we say A is a left H-module algebra (or is a left Hopf module-algebra) if (as a k-vector space) A has a left H-module structure and $m : A \otimes A \to A$ and $\eta : k \to A$ are left H-module maps in the sense that the following two properties are satisfied:

1. $hm(a \otimes b) = \sum_{i=1}^{n}(h^i_{(1)}a)(h^i_{(2)}b)$.
2. $h\eta(k) = \eta(hk)$.

Indeed, we say the last map is a left H-module map because the counit gives k a natural left H-module structure. In general, we have the following proposition:

Proposition 15.2. Any Hopf algebra H acts on on itself (as a left H-module algebra) where for all $h, g \in H$, we define

$$h \cdot g \coloneqq m((id \otimes S)(\Delta(h)(g \otimes 1))) = \sum_{i=1}^{n} h^i_{(1)} g S(h^i_{(2)}).$$ [64]

As above, and in general, we will specify the action with a $\cdot$ if there is possibility of confusion with multiplication. Let's see a few examples:

Example 15.1 (Hopf module-algebras)**.**

- Consider the Hopf algebra $H = k[G]$. Since $\Delta(h) = h \otimes h$ and $S(h) = h^{-1}$, we have $h \cdot g = hgh^{-1}$.
- Let $\mathfrak{g}$ be a Lie algebra and consider the Hopf algebra $H = \mathcal{U}(\mathfrak{g})$. If $h, g \in \mathfrak{g} \subset \mathcal{U}(\mathfrak{g})$, then recall $\Delta(h) = (h \otimes 1) + (1 \otimes h)$ and $S(h) = -h$ and similarly for g. Then $h \cdot g = hg - gh$ which the advanced reader will realize is the adjoint action on $\mathfrak{g}$.

[63]These are often called crystal bases or canonical bases in select texts.

[64]If H is commutative it can be checked that this action is trivial.

If H' is dually paired to H, then we can defined a left action of H' via the pairing $\langle \cdot, \cdot \rangle$ by

$$\phi \cdot h \coloneqq \sum_{i=1}^{n} h^i_{(1)} \langle \phi, h^i_{(2)} \rangle.$$

We'd now like to state and sketch the proof of a lemma:

Lemma 15.2. For any Lie algebra $\mathfrak{g}$, an algebra A is a left Hopf module-algebra over $\mathfrak{g}$ if and only if A has a left $\mathfrak{g}$-module structure on which elements act by derivations.

Proof sketch. For the forward direction, recall that given $g \in \mathfrak{g}$, $\Delta(g) = (g \otimes 1) + (1 \otimes g)$. If we require

$$g(ab) = \sum_{i=1}^{n} (g^i_{(1)} a)(g^i_{(2)} b),$$

this becomes $x(ab) = x(a)b + bx(a)$. Conversely, there is a small lemma which states that if A is an Hopf module satisfying the unit properties of a module and satisfies the multiplication properties on generators, then it defines a module-algebra. Using this lemma all that needs to be checked are the multiplication properties on multiplication of generators[65]. $\square$

From this lemma, we have a theorem which we will not give a proof for:

Theorem 15.5. Define an action of $\mathcal{U}(\mathfrak{sl}(2,k))$ on polynomials $p \in k[x,y]$ by

$$Xp \coloneqq x\frac{\partial p}{\partial y}, \quad Yp \coloneqq y\frac{\partial p}{\partial x}, \quad \text{and} \quad Hp \coloneqq x\frac{\partial p}{\partial x} - y\frac{\partial p}{\partial y}.$$

This makes $k[x,y]$ into a left Hopf module-algebra over $\mathcal{U}(\mathfrak{sl}(2,k))$. The left $\mathcal{U}(\mathfrak{sl}(2,k))$-submodules $k[x,y]_n$ consist of homogeneous polynomials of degree n and are isomorphic to the simple left modules $V(n)$ discussed previously.

This concludes our discussion of the representation theory of $\mathcal{U}(\mathfrak{sl}(2,k))$.

[65] We are using multiplication in two different ways here.

16 The Representation Theory of $\mathcal{U}_q(\mathfrak{sl}(2,k))$ for q Not a Root of Unity

This is going to be in the same spirit as the previous section, but the arguments and material will be more difficult. We are going to need a few preliminary statements:

Lemma 16.1. There exists a unique automorphism ω of $\mathcal{U}_q(\mathfrak{sl}(2,k))$ sending $E \mapsto F$, $F \mapsto E$, $K \mapsto K^{-1}$, and $K^{-1} \mapsto K$ such that $\omega^2 = \mathrm{id}$.

Proof sketch. Since the automorphism is defined on generators, it is unique if it exists. It's now just a short check to see that ω is compatible with all the relations defining $\mathcal{U}_q(\mathfrak{sl}(2,k))$. □

This lemma will essentially cut the workload in half as can be seen below:

Lemma 16.2. For $m \geq 0$, we have the identity

$$[E, F^m] = [m]_q F^{m-1} \frac{q^{-(m-1)}K - q^{(m-1)}K^{-1}}{q - q^{-1}}$$

where we set $[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}$.

Proof sketch. Use induction. □

There is also a dual identity which is easily proved with the help of ω:

Lemma 16.3. For $m \geq 0$, we have the identity

$$[E^m, F] = [m]_q E^{m-1} \frac{q^{(m-1)}K - q^{-(m-1)}K^{-1}}{q - q^{-1}}$$

where we set $[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}$.

Proof sketch. Apply ω to the first lemma. □

We will also need a basis theorem:

Theorem 16.1. $\mathcal{U}_q(\mathfrak{sl}(2,k))$ has $\{F^i K^j E^\ell\}_{\substack{i,\ell \geq 0 \\ j \in \mathbb{Z}}}$ as a basis.

Proof sketch. The proof is in the same spirit as for $\mathcal{U}_q(\mathfrak{b}_+)$. □

We also have a short corollary:

Corollary 16.1. $\mathcal{U}_q(\mathfrak{sl}(2,k))$ has no zero divisors.

Proof sketch. Consider the subalgebra $\mathcal{U}_0 \subset \mathcal{U}_q(\mathfrak{sl}(2,k))$ defined by

$$\mathcal{U}_0 \coloneqq \langle K, K^{-1} \rangle.$$

Observe any element of $\mathcal{U}_q(\mathfrak{sl}(2,k))$ is expressible as a linear combination of terms of the form $F^s h E^r$ where $h \in \mathcal{U}_0$ and $r, s \geq 0$. Say $u \in \mathcal{U}_q(\mathfrak{sl}(2,k))$ has leading term (r,s) if $F^s h E^r$ is a term in u and if all other terms $F^{s'} h E^{r'}$ have $s' < s$ or $s = s'$ and $r' < r$. Now show that if u has leading term (s,r) an u' has leading term (p,m), then uu' has leading term $(s+p, r+m)$. □

We now want to classify finite dimensional left modules M for $\mathcal{U}_q(\mathfrak{sl}(2,k))$. For the moment we will assume the characteristic of the base field k is not 2 and $q \in k$ is not a root of unity. We let a weight vector be an eigenvector under the action of K. If λ is an eigenvalue of M, let $M_\lambda = \{m \in M \mid Km = \lambda m\}$. The relations for $\mathcal{U}_q(\mathfrak{sl}(2,k))$ imply $EM_\lambda \subseteq M_{q^2\lambda}$ and $FM_\lambda \subseteq M_{q^{-2}\lambda}$. So the space

$$\bigoplus_{n \in \mathbb{Z}} M_{q^{2n}\lambda}$$

is a left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-submodule of M for any choice of λ. We cannot guarantee any of these summands are nonzero, but if $M_\lambda \neq 0$ for some λ and M is simple this implies

$$M = \bigoplus_{n \in \mathbb{Z}} M_{q^{2n}\lambda}$$

where all but finitely many of the summands are nonzero. We now have a proposition:

Proposition 16.1. Let M be a finite dimensional left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-module. Then there exist positive integers r and s such that $E^r M = 0$ and $F^s M = 0$.

Proof. We will prove $E^r M = 0$ as the proof for F is analogous. Let

$$M_f = \{m \in M \mid f(K)^n m = 0 \text{ for } n >> 0\}$$

where $f \in k[x]$ is an irreducible. Observe that if M_f and M_g are nonzero, then $M_f = M_g$ if and only if f is a constant multiple of g. Since M is finite dimensional

$$M = \bigoplus_{f \text{ irreducible}} M_f$$

where all but finitely many of the summands are nonzero. Given $f \in k[x]$ with $M_f \neq 0$, define $f_i(x) = f(q^i x)$. Observe f_i is irreducible as well. Since $EK = q^{-2}KE$, $Ef(K) = f_{-2}(K)E$ and induction on r shows $E^r f(K) = f_{-2r}(K)E^r$ for all $r > 0$. This implies $E^r M_f \subseteq M_{f_{-2r}}$. So it suffices to show that for all irreducible f there exists an r such that $M_{f_{-2r}} = 0$ for then (by the finite dimensionality of M) we can take the maximum such r and we will be done. Suppose $M_{f_{-2r}} \neq 0$ for all $r > 0$. Because M is finite dimensional there exists a positive integer s such that $M_{f_{-2r}} = M_{f_{-2s}}$ for all $r \geq s$. But then for $r \geq s$, f_{-2r} and f_{-2s} differ by a nonzero scalar. If the polynomials have a nonzero constant term then $M_{f_{-2r}} = M_{f_{-2s}} = M_x$ but K is invertible so $M_x = 0$ which is a contradiction. So the constant term is zero. But then they differ in the leading term by $q^{2(s-r)n}$ for some n, but this is never 1 because q is not a root of unity. Hence $M_{f_{-2r}} = 0$ for some r. □

This leads us to a corollary which we will consider as the penultimate statement for this section (with our choice of q):

Corollary 16.2. Let M be a finite dimensional left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-module. Then M is a direct sum of its weight spaces and all weights are of the form $\pm q^a$ with $a \in \mathbb{Z}$

Proof. Recall the linear algebra fact that an endomorphism of a finite dimensional vector space are diagonalizable if and only if its minimal polynomial splits into linear factors each appearing with multiplicity one. With this fact in mind, we will write down the minimal polynomial for the action of K on M. By the previous proposition, there exists $s > 0$ such that $F^s M = 0$. Recall that

$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

The relation above and the fact $F^s EM = 0$ together imply

$$F^{s-1}(FE)M = F^{s-1}\left(EF - \frac{K - K^{-1}}{q - q^{-1}}\right)M = 0.$$

If we keep moving E leftward past the remaining $s-1$ copies of F (using the relation to do so), we produce a polynomial which annihilates all of M.[66] In particular, the r-th iteration of this process produces the polynomial $h_r^{(s)}$ defined by

$$h_r^{(s)} := \prod_{j=-(r-1)}^{r-1} \frac{Kq^{r-s+j} - K^{-1}q^{-(r-s+j)}}{q - q^{-1}}.$$

[66]At the last step notice $EF^s M = 0$ (since $F^s M = 0$) so E and F are eliminated from the expression

By induction on $0 \leq r \leq s$, we find $F^{s-r}h_r^{(s)}M = 0$ for all r. Taking $r = s$, multiplying by appropriate powers of K, and removing nonzero constants, we find that the polynomial

$$\prod_{j=-(s-1)}^{s-1} (K - q^{-j})(K + q^{-j})$$

still annihilates M. Observe that it has distinct linear factors of multiplicity one. The minimal polynomial also divides it so the minimal polynomial inherits these properties and this proves the corollary. □

We would like to explore highest weight representations. These correspond to highest weight vectors which are vectors $m \in M$ such that $Em = 0$ and $Km = \lambda m$ where λ is an eigenvalue. Fixing λ, we define the universal left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-module for λ by

$$M(\lambda) \coloneqq \mathcal{U}_q(\mathfrak{sl}(2,k))/(u(K-\lambda), uE)$$

where $(u(K-\lambda), uE)$ is the left ideal generated by expressions of the form $u(K-\lambda)$ and uE for all $u \in \mathcal{U}_q(\mathfrak{sl}(2,k))$. It has $\{m_i\}_{i\in\mathbb{Z}}$ as a basis (we define $m_i \coloneqq [F^i]$) with highest weight vector $[1]$ which we call m_0. Observe that it is an infinite dimensional left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-module. If we have any M with highest weight vector m of weight λ, then there is a unique homomorphism from $M(\lambda)$ to M given by sending m_0 to m. The elements E, F, and K act on the basis for $M(\lambda)$ by

$$Em_i = \begin{cases} [i]_q \frac{\lambda q^{1-i} - \lambda^{-1} q^{i-1}}{q - q^{-1}} m_{i-1} & \text{if } i \neq 0 \\ 0 & \text{if } i = 0 \end{cases}, \quad Fm_i = m_{i+1}, \quad \text{and} \quad Km_i = \lambda q^{-2i} m_i.$$

Recall that every finite dimensional left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-module has a decomposition

$$M = \bigoplus_{\mu} M_\mu.$$ [67]

Find some weight λ with $M_\lambda \neq 0$ and $M_{q^{2n}\lambda} = 0$. Then by the universal property for left modules, there is a homomorphism

$$\varphi_\lambda : M(\lambda) \to M.$$ [68]

[67] Here we have relabeled so the sum is over all weights and not $\mathbb{Z}$. This is so we can let λ be a weight and not abuse notation.

[68] It will turn out that this homomorphism is defined up to scalar multiplication and so is independent of the choice of λ.

Moreover, φ_λ is surjective if M is simple so by the isomorphism theorems $M \cong M(\lambda)/\ker\varphi_\lambda$ with $\ker\varphi_\lambda$ a left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-submodule. So, we want to know for what values of λ does $M(\lambda)$ have left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-submodules. We have the following proposition:

Proposition 16.2. Let $\lambda \neq 0$. If $\lambda \neq \pm q^n$ for all $n >> 0$, then $M(\lambda)$ is simple. If $\lambda = \pm q^n$ for some $n \geq 0$, then $\langle \{m_i\}_{i \geq n+1} \rangle$ is the unique nontrivial left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-submodule of $M(\lambda)$ isomorphic to $M(q^{-2(n+1)}\lambda)$.

Proof. Let M' be a left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-submodule. Then as a left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-module, M' is a direct sum of its weight spaces. We know that $M(\lambda)$ has one dimensional weight spaces generated by the m_i with corresponding weights $q^{-2i}\lambda$[69]. Therefore M' is spanned by a subset of these m_i. Pick $j \geq 0$ minimal among these m_i. Since M' is closed under the action of F, M' contains $M_{q^{-2i}\lambda}$ for all $i \geq j$. If $j = 0$, then $M' = M(\lambda)$ since m_0 is the highest weight vector and we are done so assume $j > 0$. Then Em_j is a multiple of m_{j-1} in M', but we picked j minimal so $Em_j = 0$ since $EM' \subseteq M'$. Returning to the identity for the action of E in $M(\lambda)$, we find that $\lambda q^{1-j} - \lambda^{-1} q^{j-1}$. This implies $\lambda^2 = q^{2(j-1)}$ so that $\lambda = \pm q^{j-1}$. Therefore $M(\lambda)$ is simple whenever $\lambda \neq \pm q^n$, and if $\lambda = \pm q^n$ observe the basis for the proposed nontrivial left submodule and recall how the generators act to see that there is at most one left submodule. This left submodule has $Em_{n+1} = 0$. Then the universal property for left modules implies that there is a homomorphism from $M(\pm q^{-j-1})$ to $M(\lambda)$ sending m_0 to m_j. Its a quick check to show this mapping is isomorphism onto M'. ☐

We also have a quick corollary:

Corollary 16.3. For every $n \geq 0$, there are a pair of left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-submodules of dimension $n-1$. We label them $L(n,+)$ and $L(n,-)$ and they have highest weight vectors of weight q^n and $-q^n$ respectively.

Proof. Just notice $L(n,+) \cong M(q^n)/M'$ and $L(n,-) \cong M(-q^n)/M'$ where M' is the unique nontrivial left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-submodule of $M(\lambda)$. ☐

Our goal now is to use the quantum Casimir element[70] to detect finite dimensional representations. In particular, we will show that finite dimensional left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-modules are semisimple. We will accomplish this by using composition series. That is, a sequence of left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-submodules

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

[69]Here we use the fact that q is not a root of unity to guarantee distinct eigenvalues.
[70]This will be defined below.

such that M_i/M_{i-1} is simple for $1 \leq i \leq r$. We first recall the Jordan-Hölder theorem:

Theorem 16.2 (Joran-Hölder Theorem)**.** If we have some left module M and two composition series, then both series are the same length and corresponding quotients are isomorphic.

Recall that we had an element C call the Casimir element for $\mathcal{U}(\mathfrak{sl}(2,k))$. We have a corresponding element, called the quantum Casimir element, which we also denote by C. It is defined as follows:

$$C := FE + \frac{Kq + K^{-1}q^{-1}}{q - q^{-1}} = EF + \frac{Kq^{-1} + K^{-1}q}{(q - q^{-1})^2}.$$

We have a lemma describing some basic properties of the quantum Casimir element:

Lemma 16.4. Let C be the quantum Casimir element of $\mathcal{U}_q(\mathfrak{sl}(2,k))$. Then

1. C is in the center of $\mathcal{U}_q(\mathfrak{sl}(2,k))$.
2. C acts on universal left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-modules $M(\lambda)$ by scalars where the scalar is
$$\frac{\lambda q + \lambda^{-1}q^{-1}}{(q - q^{-1})^2}.$$
3. If C acts on two universal left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-modules $M(\lambda)$ and $M(\mu)$ by the same scalar, then $\lambda = \mu$ or $\lambda = q^{-2}\mu^{-1}$.

Proof sketch. The first property is proved by checking that brackets of generators with C vanish. The second is proved by noticing that it suffices to check this on a single weight space because C commutes with the m_i. Then just choose a nice weight space. The third property is a simple algebra check using the second property. □

We also get a useful corollary from the lemma:

Corollary 16.4. Let L and L' be two finite dimensional simple left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-modules. Then C acts by scalars on them, and C acts by the same scalar if and only if $L \cong L'$

Proof. From the lemma, we know C acts by scalars on $M(\lambda)$, so it acts by scalars on L since L is the homomorphic image of $M(\lambda)$ for some λ.[71] Let λ and λ' be the eigenvalues corresponding to L and L'. Plugging $\lambda = \pm q^a$ and $\lambda' = \pm q^b$ into property 2 for C above we see that if the scalars are equal then $\lambda = \lambda'$ for otherwise q is a root of unity, a contradiction. □

[71] That is, $L \cong M(\lambda)/M'$.

We now are ready to prove the main theorem of this section.

Theorem 16.3. Any finite dimensional left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-module M is semisimple.

Proof sketch. Let
$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_r = M$$
be a composition series for M. Act on the quotients M_i/M_{i-1} by C to get scalars μ_i. Then a simple induction shows that

$$\prod_{i=1}^{r}(C - \mu_i)$$

annihilates all of M.[72] This implies that M has a decomposition

$$M = \bigoplus_{\mu} M_\mu$$

where $M_\mu = \{m \in M \mid (c-\mu)^s m = 0 \text{ for } s >> 0\}$ is a generalized eigenspace. The M_μ are left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-submodules since C commutes with the action of $\mathcal{U}_q(\mathfrak{sl}(2,k))$. Therefore it suffices to prove each M_μ is semisimple. For the remainder of the proof let M be one of the M_μ. Decompose M into a composition series

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_r = M.$$

Now C acts by the scalar μ_i on M_i/M_{i-1}, so $(c-\mu_i)^s$ annihilates M for some $s >> 0$. Therefore all the μ_i are equal and we denote their common value by μ. This implies $M_i/M_{i-1} = L(n,\epsilon)$ for some $n \geq 0$, weight ϵ, and all $1 \leq i \leq r$. Our guess is that M decomposes as follows
$$M = \bigoplus_{i=1}^{r} L(n,\epsilon).$$
To check this, decomposes M into weight spaces (according to the action of K not C) so we may write
$$M = \bigoplus_{\nu} M_\nu.$$
Since dimension respects composition series we have that

$$\dim(M_\nu) = \sum_{i=1}^{r} \dim(M_i/M_{i-1})_\nu = r\dim(L(n,\epsilon))_\nu.$$

[72]Notice $(c-\mu_i)(M_i/M_{i-1}) = 0$ so $(c-\mu_i)M_i \subset M_{i-1}$.

But $\dim(L(n,\epsilon))_\nu = 1$ so $\dim(M_\nu) = r$ with highest weight $\lambda = q^n\epsilon$. So let $v_{i1 \leq i \leq r}$ be a basis. Then

$$M = \sum_{i=1}^{r} \mathcal{U}_q(\mathfrak{sl}(2,k))v_i,$$

and comparing dimensions shows that this is a direct sum. $\square$

17 The Representation Theory of $\mathcal{U}_q(\mathfrak{sl}(2,k))$ for q a Root of Unity

Having finished the representation theory of $\mathcal{U}_q(\mathfrak{sl}(2,k))$ when q is not a root of unity, we will turn to the theory when q is a root of unity. We will still assume the characteristic of k is not 2. Assume q is an ℓ-th root of unity. This implies $[\ell]_q = 0$ and $[i]_q! = 0$ for $i \geq \ell$. We have a short proposition:

Proposition 17.1. The elements E^ℓ, F^ℓ, K^ℓ, and $K^{-\ell}$ are all in the center of $\mathcal{U}_q(\mathfrak{sl}(2,k))$.

Proof sketch. As usual, just check this on generators using the relations for the algebra. □

This means we cannot use the quantum Casimir element to detect left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-modules like last time. However, we are able to put some additional restrictions on ℓ. If ℓ is even, write $\ell = 2\ell'$. Then $[\ell']_q = 0$ since $q^{\ell'} - q^{-\ell'} = 0$. By the same argument for the proposition above we can prove that $E^{\ell'}$, $F^{\ell'}$, $K^{\ell'}$, and $K^{-\ell'}$ are all central. So we may assume ℓ is at least 3 and odd.[73] For an arbitrary eigenvalue λ, recall the universal left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-module $M(\lambda)$ and observe that this construction did not depend on if q was a root of unity or not. However, since q is a root of unity some of the actions for $M(\lambda)$ simplify:

$$Em_\ell = 0 \quad \text{and} \quad Km_\ell = \lambda m_\ell.$$

This tells us that m_ℓ is a highest weight vector. Hence any vector of the form $m_\ell - bm_0$ with $b \in k$ is a highest weight vector. We now construct a module which will be useful in analyzing the representation theory of $\mathcal{U}_q(\mathfrak{sl}(2,k))$. Fixing some λ as above, for $b \in k$ define the a left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-module $Z_b(\lambda)$ by

$$Z_b(\lambda) \coloneqq M(\lambda)/(m_\ell - bm_0)$$

where $(m_\ell - bm_0)$ is the ideal generated by $m_\ell - bm_0$. Since $K(m_\ell - bm_0) = \lambda(m_\ell - bm_0)$ and $E(m_\ell - bm_0) = 0$, we conclude that this ideal is spanned by elements $F^i(m_\ell - bm_0) = m_{i+1} - bm_i$ for $i \in \mathbb{Z}_+$. This implies $Z_b(\lambda)$ has $\{m_j\}_{0 \leq j < \ell}$ as a basis.[74] The elements E and F act on the basis for $Z_b(\lambda)$ (in $Z_b(\lambda)$) by

$$Em_i = \begin{cases} [i]_q \frac{\lambda q^{1-i} - \lambda^{-1} q^{i-1}}{q - q^{-1}} m_{i-1} & \text{if } i \neq 0 \\ 0 & \text{if } i = 0 \end{cases} \quad \text{and} \quad Fm_i = \begin{cases} m_{i+1} & \text{if } i \neq \ell - 1 \\ bm_0 & \text{if } i = \ell - 1 \end{cases}$$

[73]For the general case just replace ℓ with 2ℓ.

[74]We write m_j instead of $[m_j]$ to simply notation.

where the action by K is the same as for $M(\lambda)$ except the power of λ is taking modulo ℓ. Notice that no power of F is annihilates all of $Z_b(\lambda)$.[75] Another way of phrases the last statement is that F is not a nilpotent operator on $Z_b(\lambda)$.[76] Also notice that λ need not be of the form $\pm q^n$ for some $n \in \mathbb{Z}$.

We would now like to know when $Z_b(\lambda)$ is simple. We have a proposition for this:

Proposition 17.2. Then we have the following statements:

1. If $b \neq 0$ or $\lambda^{2\ell} \neq 1$, then $Z_b(\lambda)$ is simple.
2. If $b = 0$ and $\lambda = \pm q^n$ where $0 \leq n < \ell$, then $Z_b(\lambda)$ is simple if $n = \ell - 1$ or either has a unique nonzero left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-submodule of the form $\langle m_j \mid n < j < \ell \rangle$ if $n < \ell - 1$.

Proof sketch. We will prove both statements at the same time. Let M be a nonzero proper left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-submodule. We know $Z_b(\lambda)$ is a direct sum of weight spaces, and M is closed under the action of of K so M is the direct sum of the weight spaces for $Z_b(\lambda)$ each intersected with M. Moreover, the weight vectors are corresponding m_i. Choose j minimal so that $m_j \in< M$. Since M is closed under the action of F, the weight vectors in M are precisely $m_i = F^{i-j} m_j$ for $j \leq i < \ell$. We can also assume $j > 0$ for otherwise M is not proper. Recall $Fm_{\ell-1} = bm_0$, but we have j minimal so $b = 0$. This is then a necessary condition to not be simple. On the other hand, Em_j is a multiple of $m_{j-1} \notin M$. Hence $Em_j = 0$ and this occurs if and only if $\lambda q^{1-j} - \lambda^{-1} q^{j-1} = 0$. Hence $\lambda = \pm q^{j-1}$; this is the second necessary condition for M to not be simple. Now its an easy check to show that if $b = 0$ and $\lambda = \pm q^n$ where $0 \leq n < \ell$ then $\langle m_j \mid n < j < \ell \rangle$ is the unique nonzero left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-submodule. $\square$

Its not hard to check that if $0 \leq n < \ell$ then the unique left submodule is isomorphic to $L(n,+)$ or $L(n,-)$. Also notice that $Z_0(\pm q^n)$ since it has a unique nonzero proper left submodule.[77] We also have a final proposition which will conclude this section:

Proposition 17.3. There are no finite dimensional simple left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-modules of dimension greater than ℓ.

Proof sketch. We will only sketch the proof of this for the case where k is algebraically closed despite the proposition not needing this additional assumption. If V is a finite

[75] This is unlike the case where q was not a root of unity.

[76] It acts in more of a cyclic natural by sending $m_{\lambda-1}$ to bm_0.

[77] It if split into a direct sum then there would be two left submodules.

dimensional simple left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-module with $\dim(V) > \ell$, then either V has a lowest weight vector or it does not. If $v \in V$ is a lowest weight vector then we claim that $\langle v, Ev, \ldots, E^{\ell-1}v \rangle$ is a nonzero proper left submodule of V contradicting V being simple. It will be useful to observe $E^{\ell}v = \lambda v$ for some nonzero scalar λ because $E^{\ell}v$ is central. If V does not have a lowest weight vector, then there exists a $v \in V$ with $Kv = \lambda v$ for some nonzero scalar λ since k is assumed algebraically closed. We claim $\langle v, Fv, \ldots, F^{\ell-1}v \rangle$ is a nonzero proper left submodule. □

This concludes our discussion of the representation theory of $\mathcal{U}_q(\mathfrak{sl}(2,k))$ for q a root of unity.[78]

[78] We have not given a complete classification, but it is possible to do so.

18 Example of a Quantum Group

For the remainder of this section let q be a root of unity of order ℓ with $\ell \geq 3$ odd (as before). Having answered the first three questions in section 14, we'd like to answer the fourth: is $\mathcal{U}_q(\mathfrak{sl}(2,k))$ a quantum group and if so can we determine a method to compute R? Sadly, $\mathcal{U}_q(\mathfrak{sl}(2,k))$ is not a quantum group since R would end up being an infinite sum of elements in $\mathcal{U}_q(\mathfrak{sl}(2,k)) \otimes \mathcal{U}_q(\mathfrak{sl}(2,k))$ and hence not an element of the tensor product. Not all our work is lost however since a finite dimensional quotient of $\mathcal{U}_q(\mathfrak{sl}(2,k))$ will turn out to be a quantum group. In the following we will construct this quotient and prove it is a quantum group.
Recall that we had a quantum Casimir element C in the center of $\mathcal{U}_q(\mathfrak{sl}(2,k))$. We also had central elements E^ℓ, F^ℓ, K^ℓ, and $K^{-\ell}$. Define the quotient $\overline{\mathcal{U}}_q(\mathfrak{sl}(2,k))$ by

$$\overline{\mathcal{U}}_q(\mathfrak{sl}(2,k)) = \mathcal{U}_q(\mathfrak{sl}(2,k))/(E^\ell, F^\ell, K^\ell - 1)$$

where $(E^\ell, F^\ell, K^\ell - 1)$ is the ideal generated by E^ℓ, F^ℓ, and $K^\ell - 1$. Recall that $Z_b(\lambda)$ was a simple left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-module (under some mild constraints on b and λ). More generally, we can give an abstract left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-module structure on basis generators $m_0, \ldots, m_{\ell-1}$ by

$$Em_i = \begin{cases} \left([i]_q \frac{\lambda q^{1-i} - \lambda^{-1} q^{i-1}}{q - q^{-1}} + ab\right) m_{i-1} & \text{if } i \neq 0 \\ am_{\ell-1} & \text{if } i = 0 \end{cases} \quad \text{and} \quad Fm_i = \begin{cases} m_{i+1} & \text{if } i \neq \ell - 1 \\ bm_0 & \text{if } i = \ell - 1 \end{cases}$$

where the action by K is the same as for $Z_b(\lambda)$, and $a, b \in k$. Call these left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-modules $Z_{a,b}(\lambda)$. It is a natural question to ask if $Z_{a,b}(\lambda)$ is a natural quotient of $M(\lambda)$ where $Z_b(\lambda) = Z_{0,b}(\lambda)$. We will answer this question soon.
We will introduce a final family of ℓ-dimensional left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-modules called $\tilde{Z}_c(\mu)$ where $\mu \neq 0$ and $c \in k$. They will have basis generators $m_0, \ldots, m_{\ell-1}$ with action given by

$$Em_i = \begin{cases} m_{i+1} & \text{if } i \neq 0 \\ cm_0 & \text{if } i = \ell - 1 \end{cases} \quad \text{and} \quad Fm_i = \begin{cases} [i]_q \frac{\mu^{-1} q^{1-i} - \mu q^{i-1}}{q - q^{-1}} m_{i-1} & \text{if } i \neq 0 \\ 0 & \text{if } i = 0 \end{cases}$$

where the action by K is the same as for $Z_b(\lambda)$ with μ in place of λ and the sign in the power is positive. Having defined these families of left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-modules, we may now state a theorem:

Theorem 18.1. Every l-dimensional simple left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-module is one of the following three types:

1. $Z_{a,b}(\lambda)$ with $b \neq 0$.
2. $Z_{a,0}(\lambda)$ with $\lambda \neq \pm q^n$ where $0 \leq n < \ell - 1$.
3. $\tilde{Z}_c(\pm q^n)$ with $c \neq 0$ and $0 \leq n < \ell - 1$.

We also state two theorems about $\overline{\mathcal{U}}_q(\mathfrak{sl}(2,k))$ which we will not prove. The first is:

Theorem 18.2. $\overline{\mathcal{U}}_q(\mathfrak{sl}(2,k))$ has $\{E^i F^j K^m\}_{0\leq i,j,m\leq \ell-1}$ as a basis.

The second is:

Theorem 18.3. The simple left $\overline{\mathcal{U}}_q(\mathfrak{sl}(2,k))$-modules are those for which E^ℓ, F^ℓ, and $K^\ell - 1$ act by 0 on the corresponding left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-module. In particular, they are of the form $L(n,\epsilon)$ with $0 \leq n \leq \ell - 1$ and $\epsilon = \pm 1$ or $Z_{0,0}(\pm q^{-1})$.

We now have all the machinery necessary to show that $\overline{\mathcal{U}}_q(\mathfrak{sl}(2,k))$ is a quantum group.

Theorem 18.4. $\overline{\mathcal{U}}_q(\mathfrak{sl}(2,k))$ is a quantum group with

$$R = R_K e_{q^{-2}}^{(q-q^{-1})(E\otimes F)}$$

where we define

$$R_K := \frac{1}{\ell}\sum_{a,b=0}^{\ell-1} q^{-2ab}(K^a \otimes K^b) \quad \text{and} \quad e_{q^{-2}}^{(q-q^{-1})(E\otimes F)} := \sum_{r=0}^{\ell-1} \frac{(q-q^{-1})^r (E^r \otimes F^r)}{[r]_{q^{-2}}!}$$

and set $[r]_{q^{-2}} = \frac{1-q^{-2r}}{1-q^{-2}}$.

Proof sketch. We first need to show $\overline{\mathcal{U}}_q(\mathfrak{sl}(2,k))$ is a Hopf algebra and then show that R satisfies all the necessary properties for $\overline{\mathcal{U}}_q(\mathfrak{sl}(2,k))$ to be a quantum group. Observe that it is enough to show $\overline{\mathcal{U}}_q(\mathfrak{sl}(2,k))$ is compatible with the comultiplication Δ from $\mathcal{U}_q(\mathfrak{sl}(2,k))$ to show it is a Hopf algebra. By induction and the q-binomial theorem we have

$$\Delta(E^m) = \sum_{r=0}^{m} \begin{bmatrix} r \\ m \end{bmatrix}_{q^{-2}} (E^r \otimes K^r E^{m-r}).$$

Since $\Delta(E^\ell)$ involves $[\ell]_{q^{-w}} = 0$ we have

$$\Delta(E^\ell) = (E^\ell \otimes K^\ell) + (1 \otimes E^\ell),$$

and so we may set $E^\ell = 0$ and it is compatible with Δ. A similar argument holds for F^ℓ, and clearly we can set $K^\ell = 1$. It follows that $\overline{\mathcal{U}}_q(\mathfrak{sl}(2,k))$ is a Hopf algebra. We now need to show R has the desired properties for $\overline{\mathcal{U}}_q(\mathfrak{sl}(2,k))$ to be a quantum group. It can be checked directly that $(S \otimes \mathrm{id})(R)$ is the inverse to R proving R is invertible.[79] We now show $\tau \circ \Delta = R\Delta R^{-1}$. It suffices to check this on generators, and for E this identity is equivalent to

$$R\Delta(E) = (\tau \circ \Delta(E))R.$$

Recall that $\Delta(E) = (E \otimes K) + (1 \otimes E)$ so we want to show that

$$R((E \otimes K) + (1 \otimes E)) = ((E \otimes 1) + (K \otimes E))R.$$

We need a small claim to make the identity above easier to show. In particular, we claim

$$R_K(1 \otimes E) = (K \otimes E)R_K.$$

This is proved by using $KE = q^2EK$ and reparameterizing the sum in $R((E \otimes K) + (1 \otimes E))$. There is a similar claim for passing R_K past $E \otimes K$. Now all that remains is to pass $e_{q^{-2}}^{(q-q^{-1})(E\otimes F)}$ past $((E \otimes K) + (1 \otimes E))$ (recall $R = R_K e_{q^{-2}}^{(q-q^{-1})(E\otimes F)}$) and check that taking products for the R_K and $e_{q^{-2}}^{(q-q^{-1})(E\otimes F)}$ pieces gives $((E \otimes 1) + (K \otimes E))R$. This is just an exercise in q-calculus. The argument is analogous for F, and the case is even easier for K and K^{-1}. We are left to check that R satisfies the identities

$$(\Delta \otimes \mathrm{id})(R) = R_{(1,3)}R_{(2,3)} \quad \text{and} \quad (\mathrm{id} \otimes \Delta)(R) = R_{(1,3)}R_{(1,2)}.$$

We will only prove the first of these two identities since the argument will be analogous for the second. To do this, we first need to analyze R_K in the Hopf subalgebra H_K (of $\overline{\mathcal{U}}_q(\mathfrak{sl}(2,k))$) generated by K. It's an easy check to show that H_K is both commutative and cocommutative. A general element of $H_K \otimes H_K$ has the form

$$\sum_{a,b=0}^{\ell-1} \alpha_{a,b}(K^a \otimes K^b).$$

Consider the identity

$$(\Delta \otimes \mathrm{id})(R) = R_{(1,3)}R_{(2,3)}$$

[79] If you are concerned how we came up with this inverse, if $(\epsilon \otimes \mathrm{id})(R) = 1$, then we can deduce $R^{-1} = (S \otimes \mathrm{id})(R)$. Indeed, it can be checked in this case that $(\epsilon \otimes \mathrm{id})(R) = 1$.

in H_K for some general $R \in H_K \otimes H_K$. If we compute the left-hand side for a general element, we find

$$\sum_{a,b,c,d=0}^{\ell-1} \alpha_{a,b}\alpha_{c,d}(K^a \otimes K^c \otimes K^{b+d}).$$

Making the substitution $d' = b + d$ we can rewrite the expression above as

$$\sum_{a,b,c,d'=0}^{\ell-1} \alpha_{a,b}\alpha_{c,d'-b}(K^a \otimes K^c \otimes K^{d'}).$$

Similarly, computing the right-hand sides for an general element gives

$$\sum_{a,d'=0}^{\ell-1} \alpha_{a,d'}(K^a \otimes K^a \otimes K^{d'}).$$

So if we want the identity to hold, then its sufficient to have

$$\sum_{b=0}^{\ell-1} \alpha_{a,b}\alpha_{c,d'-b} = \begin{cases} \alpha_{a,d'} & \text{if } a = c \\ 0 & \text{if } a \neq c \end{cases}.$$

Now observe that for R_K we have $\alpha_{a,b} = q^{-2ab}/\ell$ and a quick check shows R_K satisfies the necessary sum condition above for H_K. There is an analogous condition for the other identity and R_K satisfies this as well. Now we analyze $e_{q^{-2}}^{(q-q^{-1})(E\otimes F)}$. Apply the q-binomial theorem to get

$$(\Delta \otimes \mathrm{id})\left(e_{q^{-2}}^{(q-q^{-1})(E\otimes F)}\right) = e_{q^{-2}}^{(q-q^{-1})((E\otimes K\otimes F)+(1\otimes E\otimes F))}.$$

We can now compute $(\Delta \otimes \mathrm{id})(R)$ in $\overline{\mathcal{U}}_q(\mathfrak{sl}(2,k))$:

$$\begin{aligned}(\Delta \otimes \mathrm{id})(R) &= (\Delta \otimes \mathrm{id})R_K e_{q^{-2}}^{(q-q^{-1})(E\otimes F)} \\ &= (R_K)_{(1,3)}(R_K)_{(2,3)} e_{q^{-2}}^{(q-q^{-1})((E\otimes K\otimes F)+(1\otimes E\otimes F))} \\ &= (R_K)_{(1,3)}(R_K)_{(2,3)} e_{q^{-2}}^{(q-q^{-1})(E\otimes K\otimes F)} e_{q^{-2}}^{(q-q^{-1})(1\otimes E\otimes F)} \\ &= (R_K)_{(1,3)} e_{q^{-2}}^{(q-q^{-1})(E\otimes 1\otimes F)} (R_K)_{(2,3)} e_{q^{-2}}^{(q-q^{-1})(1\otimes E\otimes F)} \\ &= R_{(1,3)}R_{(2,3)}\end{aligned}$$

where the third equality follows from a q-calculus check that

$$e_{q^{-2}}^{(q-q^{-1})(A+B)} = e_{q^{-2}}^{(q-q^{-1})A} e_{q^{-2}}^{(q-q^{-1})B}$$

for arbitrary tensors A and B, and the fourth equality follows by using $R_K(K \otimes F) = (1 \otimes F)R_K$. As we noted above, the other identity is proved similarly. This finishes the proof that $\overline{\mathcal{U}}_q(\mathfrak{sl}(2,k))$ is a quantum group. $\square$

This also answers the first part of our fourth question. The second part of our fourth question, and our last question, will both be answered in the following sections.

19 Quantum Doubles and Bicrossed Products

We would now like to describe quantum doubles.[80] The idea is to construct a quantum group given a finite dimensional Hopf algebra and its dual. To motivate the following, this construction will explain how to naturally choose R for the quasitriangular structure of $\overline{\mathcal{U}}_q(\mathfrak{sl}(2,k))$.

We start with a finite dimensional Hopf algebra H. Recall that there is a natural Hopf algebra structure on the dual space H^* induced by the evaluation map $\langle\cdot,\cdot\rangle : H^* \otimes H \to k$. We will use H, H^*, and the evaluation map to construct the quantum double $D(H)$. The quantum double will turn out to be automatically quasi-triangular and self-dual. We often denoted the quantum double of H by $(H^{\mathrm{op}})^* \bowtie H$ as well. Let's explain this notation a little more. For an arbitrary algebra A, we write A^{op} for the opposite algebra of A. The multiplication in the opposite algebra is defined by

$$m^{\mathrm{op}}(a \otimes b) = m(b \otimes a).$$

In other words, the multiplication is reversed in A^{op}. The unit is the same unit as A. If we have an arbitrary Hopf algebra H and S is bijective (which always happens if H is finite dimensional) then H^{op}, H with the opposite algebra structure and same coalgebra structure, is a bialgebra and a Hopf algebra if we define

$$S^{\mathrm{op}} := S^{-1}.$$

So by $(H^{\mathrm{op}})^*$ we mean the dual of the opposite Hopf algebra H. Now $\bowtie$ is called the bicrossed product and we will define the quantum double of H via the bicrossed product, but before we do it will be useful to define the bicrossed product of groups.

As the notation suggests, $\bowtie$ should be connected to the semidirect product of groups. Recall that if G is a group, $H \trianglelefteq G$, and $K \leq G$, then we write $G = H \rtimes K$ and say G is the semidirect product of H and K if any of the following equivalent conditions holds

- If $g \in G$, then $g = hk$ for unique $h \in H$ and $k \in K$.
- If $g \in G$, then $g = k'h'$ for unique $k' \in K$ and $h' \in H$.
- $G = H \times K$ as a set where $H \cap K = \{1\}$.

[80]This construction is sometimes called the Drinfeld double.

Observe that if $g_1 = h_1k_1$ and $g_2 = h_2k_2$, then

$$\begin{aligned} g_1g_2 &= h_1k_1h_2k_2 \\ &= h_1k_1h_2k_1^{-1}k_1k_2 \\ &= h_1h_2^{k_1}k_1k_2 \end{aligned}$$

where $h_2^{k_1}$ denotes conjugation by k_1 applied to h_2 and $h_2^{k_1} \in H$ since H is normal. In other words, G is isomorphic to the group with underlying set $H \times K$ and multiplication defined by

$$(h_1, k_1) \cdot (h_2, k_2) := (h_1h_2^{k_1}, k_1k_2).$$

The identity is $(1_H, 1_K)$. This semidirect product is often called the inner semidirect product. We can define a more general construction called the outer semidirect product as follows: given two arbitrary groups H and K and a group homomorphism $\phi : K \to \mathrm{Aut}(H)$, we define the outer semidirect product of H and K with respect to ϕ, denoted $H \rtimes_\phi K$, as the group with underlying set $H \times K$ and multiplication defined by

$$(h_1, k_1) \cdot (h_2, k_2) := (h_1\phi(k_1)(h_2), k_1k_2).$$

The identity is again $(1_H, 1_K)$. Observe that if $H \trianglelefteq G$, $K \leq G$, and ϕ is given by conjugation, then $H \rtimes_\phi K = H \rtimes K$. So the outer semidirect product generalizes the inner semidirect product. To define the bicrossed product, we need to generalize this construction even further. If H and K are arbitrary groups with left and right actions $\alpha : K \times H \to H$ and $\beta : K \times H \to K$ respectively and satisfying the following properties:

1. $\alpha(k, h_1h_2) = \alpha(k, h_1)\alpha(\beta(k, h_1), h_2)$.
2. $\alpha(k, 1_H) = 1_H$.
3. $\beta(k_1k_2, h) = \beta(k_1, \alpha(k_2, h))\beta(k_2, h)$.
4. $\beta(1_K, h) = 1_K$.

then we say K and H are a pair of matched groups with respect to α and β or simply a matched pair.[81] For a matched pair K and H (with respect to α and β) we define

[81] Notice that we reference K first by saying K and H are a matched pair. This is done to emphasis that α and β are maps from $K \times H$.

their bicrossed product, denoted $H_\alpha \bowtie_\beta K$ or sometimes just $H \bowtie K$, as the group with underlying set $H \times K$, and multiplication defined by

$$(h_1, k_1) \cdot (h_2, k_2) := (h_1 \alpha(k_1, h_2), \beta(k_1, h_2) k_2).$$

The unit is again $(1_H, 1_K)$. Observe that if β is the trivial action and given a group homomorphism $\phi : K \to \mathrm{Aut}(H)$ we set $\alpha(k, h) = \phi(k)(h)$, then K and H are a matched pair. In particular, $H_\alpha \bowtie_\beta K = H \rtimes_\phi K$. So, the bicrossed product generalizes the outer semidirect product.

We're going to generalize bicrossed products to Hopf algebras. Recall that we have already discussed left Hopf module-algebras. To recapitulate, if H is a Hopf algebra and A is an algebra (both over k), then we say that A is a left H-module algebra if (as a k-vector space) A has a left H-module structure and $m : A \otimes A \to A$ and $\eta : k \to A$ are left H-module maps in the sense that the following two properties are satisfied:

1. $hm(a \otimes b) = \sum_{i=1}^{n} (h^i_{(1)} a)(h^i_{(2)} b)$.
2. $h\eta(k) = \eta(hk)$.

We have a similar definition for right Hopf module-coalgebras. If H is a Hopf algebra and C is a coalgebra (both over k) we say C is a right H-module coalgebra (or is a right Hopf module-coalgebra) if (as a k-vector space) C has a right H-module structure and $\Delta : C \to C \otimes C$ and $\epsilon : C \to k$ are right H-module maps in the sense that the following two properties are satisfied:

1. $\Delta(a)h = \sum_{i=1}^{n} (a^i_{(1)} h)(a^i_{(2)} h)$.
2. $\epsilon(a)h = \epsilon(ha)$.

We can now defined bicrossed products for Hopf algebras. If H and K are two Hopf algebras with maps $\alpha : K \otimes H \to H$ and $\beta : K \otimes H \to K$ such that α makes H into a left K-module coalgebra, β makes K into a right H-module coalgebra, and the following properties are satisfied:

1. $\alpha(k \otimes h_1 h_2) = \sum_{i=1}^{n} \alpha(k^i_{(1)} \otimes h^i_{1_{(1)}}) \alpha(\beta(h^i_{1_{(2)}} \otimes k^i_{(2)}) \otimes h_2)$.
2. $\alpha(k \otimes 1) = \epsilon_K(k)$.
3. $\beta(k_1 k_2 \otimes h) = \sum_{i=1}^{n} \beta(k_1 \otimes \alpha(h^i_{(2)} \otimes k^i_{2_{(2)}})) \beta(k^i_{2_{(1)}} \otimes h^i_{(1)})$.
4. $\beta(h \otimes 1) = \epsilon_H(h)$.

5. $\sum_{i=1}^{n} \beta(h^i_{(1)} \otimes k^i_{(1)}) \otimes \alpha(k^i_{(2)} \otimes h^i_{(2)}) = \sum_{i=1}^{n} \beta(h^i_{(2)} \otimes k^i_{(2)}) \otimes \alpha(k^i_{(1)} \otimes h^i_{(1)})$.

then we say K and H are a matched pair with respect to α and β or simply a matched pair.[82] We define the bicrossed product of a matched pair K and H (with respect to α and β), denoted $H_\alpha \bowtie_\beta K$ or sometimes just $H \bowtie K$, as the unique Hopf algebra[83] with underling k-vector space $H \otimes K$ and with the following structure: its algebra structure is given by defining the multiplication $m_\bowtie$ as

$$m_\bowtie((h_1 \otimes k_1) \otimes (h_2 \otimes k_2)) := \sum_{i=1}^{n} h_1 \alpha(k^i_{1_{(1)}} \otimes h^i_{2_{(1)}}) \otimes \beta(h^i_{2_{(2)}} \otimes k^i_{1_{(2)}}) k_2,$$

where the unit $\eta_\bowtie$ is defined by

$$\eta_\bowtie(a) := \eta_H(a) \otimes \eta_K(a).$$

Its coalgebra structure is given by defining the comultiplication $\Delta_\bowtie$ as

$$\Delta_\bowtie(h \otimes k) := \sum_{i=1}^{n} (h^i_{(1)} \otimes k^i_{(1)}) \otimes (h^i_{(2)} \otimes k^i_{(2)}),$$

where the counit $\epsilon_\bowtie$ is defined by

$$\epsilon_\bowtie(h \otimes k) = \epsilon_H(h) \otimes \epsilon_K(k).$$ [84]

We avoid defining the antipode $S_\bowtie$, but rest assured it is defined by using S_H and S_K. This bicrossed product respects the bicrossed product for groups in the sense that if H and K are finite groups and we consider the Hopf algebras $k[H]$ and $k[K]$, then $k[H \bowtie K] \cong k[H] \bowtie k[G]$ as Hopf algebras.

We are now ready to define the quantum double. If H is a finite dimensional Hopf algebra then H and $(H^{\text{op}})^*$ are a matched pair with respect to the actions

$$\alpha : H \otimes (H^{\text{op}})^* \to (H^{\text{op}})^* \qquad (h \otimes \phi) \mapsto \alpha(h)\phi(x) := \sum_{i=1}^{n} \phi(S^{-1}(h^i_{(2)}) x h^i_{(1)})$$

and

$$\beta : H \otimes (H^{\text{op}})^* \to (H^{\text{op}})^* \qquad (h \otimes \phi) \mapsto \beta(\phi)h := \sum_{i=1}^{n} \phi(S^{-1}(h^i_{(3)}) h^i_{(1)}) h^i_{(2)}$$

[82]Notice again that we say K and H are a matched pair to emphasis α and β have domain $K \otimes H$.
[83]We will not prove existence or uniqueness although they are not hard to prove.
[84]As usual we make the identification $k \cong k \otimes k$.

where the terms $h^i_{(3)}$ appear because we have used comultiplication twice on the $h^i_{(2)}$ and relabeled the resulting terms by $h^i_{(2)}$ and $h^i_{(3)}$ (this is abusive). Then the quantum double of H is defined to be $(H^{\text{op}})^*{}_\alpha\!\bowtie_\beta H$ or simply $(H^{\text{op}})^* \bowtie H$. We note that α and β in the definition of the quantum dual are natural in that they arise from natural actions defined using the dual pairing between H and H^* (in this case the evaluation map).

The first property of quantum doubles we will prove is that they are somewhat trivial when H is cocommutative:

Proposition 19.1. If H is a finite dimensional cocommutative Hopf algebra, then in $(H^{\text{op}})^* \bowtie H$, β acts trivially, i.e.,

$$\beta(\phi \otimes h) = \epsilon^{\text{op}^*}(\phi)h.$$

Proof. This is just a computation of the β action.

$$\begin{aligned}
\beta(\phi \otimes h) &= \sum_{i=1}^{n} \phi(S^{-1}(h^i_{(3)})h^i_{(1)})h^i_{(2)} && \text{(by definition of } \beta\text{)} \\
&= \sum_{i=1}^{n} \phi(S^{-1}(h^i_{(3)})h^i_{(2)})h^i_{(1)} && \text{(cocommutativity of } H\text{)} \\
&= \sum_{i=1}^{n} \phi(\epsilon(h^i_{(2)}))h^i_{(1)} && \text{(antipode property of } S^{-1}\text{)} \\
&= \sum_{i=1}^{n} \phi(1)\epsilon(h^i_{(2)})h^i_{(1)} && \text{(linearity)} \\
&= \epsilon^{\text{op}^*}(\phi)h && \text{(definition of the pairing and counit property)}
\end{aligned}$$

□

In the case that β acts trivially (so in particular if H is cocommutative), $(H^{\text{op}})^* \bowtie H$ is isomorphic to a structure called the crossed product which is akin to the semidirect product of groups.

We would now like to prove $D(H)$ is always quasitriangular. To do so we need a candidate element $R \in D(H) \otimes D(H)$. Luckily, the desired candidate is of a particularly nice form. If I is an indexing set where $\{e_i\}_{i\in I}$ is a basis for H and $\{e^i\}_{i\in I}$ is the dual basis for H^*, then $R \in D(H) \otimes D(H)$ will be

$$R = \sum_{i\in I}(1 \otimes e_i) \otimes (e^i \otimes 1).$$

Before proving that this choice of R makes $D(H)$ into a quantum group, we are going to give some reasoning for why such an R is natural. Recall that if U and V are finite dimensional vector spaces then there is a natural isomorphism

$$\lambda_{U,V} : V \otimes U^* \to \mathrm{Hom}(U, V) \qquad v \otimes \phi \mapsto (u \mapsto \phi(u)v).$$

In particular, if $U = V = H$ where H is a finite dimensional Hopf algebra, then we have an isomorphism $\lambda_{H,H} : H \otimes H^* \to \mathrm{End}(H)$. Set $\rho = \lambda_{H,H}^{-1}(\mathrm{id})$. By choosing a basis $\{e_i\}_{i\in I}$ for H and letting $\{e^i\}_{i\in I}$ be the dual basis for H^*, we have

$$\rho = \sum_{i\in I} e_i \otimes e^i.$$

If we define $R \in D(H) \otimes D(H)$ by

$$R := (\iota_H \otimes \iota_{(H^{\mathrm{op}})^*})(\rho)$$

where ι_H and $\iota_{(H^{\mathrm{op}})^*}$ are the natural inclusions of H and $(H^{\mathrm{op}})^*$ into $D(H)$ respectively, then in terms of the bases we find

$$R = \sum_{i\in I} (1 \otimes e_i) \otimes (e^i \otimes 1)$$

which was our original choice for R. We can now sketch the proof of the following theorem:

Theorem 19.1. Let H be a finite dimensional Hopf algebra. The quantum double $D(H)$ is quasitriangular with

$$R = \sum_{i\in I} (1 \otimes e_i) \otimes (e^i \otimes 1).$$

Proof sketch. The idea is to use the dual pairing between H and H^* to express the properties $R \in D(H) \otimes D(H)$ needs to satisfy in terms of H. We will sketch the proof that

$$(\Delta \otimes \mathrm{id})(R) = R_{(1,3)} R_{(2,3)}$$

as the other properties are checked in the same way. To prove this property we can construct a dual pairing between $D(H)^{\otimes 3}$ and its dual from the dual paring between H and H^*. This dual pairing on $D(H)^{\otimes 3}$ is nondegenerate so

$$(\Delta \otimes \mathrm{id})(R) = R_{(1,3)} R_{(2,3)}$$

is equivalent to showing that $\langle(\Delta \otimes \mathrm{id})(R), \theta\rangle = \langle R_{(1,3)} R_{(2,3)}, \theta\rangle$ for an arbitrary element θ in the dual of $D(H)^{\otimes 3}$. Letting $\theta = \alpha \otimes \tau \otimes b \otimes \mu \otimes c \otimes \nu$ be an arbitrary element of $D(H)^{\otimes 3}$, its not difficult to check that $\langle(\Delta\otimes\mathrm{id})(R), \theta\rangle$ and $\langle(R_{(1,3)} R_{(2,3)}, \theta\rangle$ are both equal to

$$\epsilon(a)\epsilon(b)\nu(1) \sum_{i=1}^{n} \tau(c^i_{(1)})\mu(c^i_{(2)}).$$

$\square$

20 $\overline{\mathcal{U}}_q(\mathfrak{b}_+)$ and $\overline{\mathcal{U}}_q(\mathfrak{sl}(2,k))$

We're going to discuss a Hopf algebra associated to $\overline{\mathcal{U}}_q(\mathfrak{sl}(2,k))$ and how its quantum double resembles $\overline{\mathcal{U}}_q(\mathfrak{sl}(2,k))$. Recall that $\mathfrak{b}_+$ is a subalgebra of $\mathfrak{sl}(2,k)$. We define $\overline{\mathcal{U}}_q(\mathfrak{b}_+)$ to be the Hopf subalgebra of $\mathcal{U}_q(\mathfrak{sl}(2,k))$ with k-vector space basis $\{E^m K^n\}_{0\leq m,n\leq \ell-1}$ and with relation $KEK^{-1} = q^2 E$.

Before moving on, we'd like to discuss a subtly with the relation $KEK^{-1} = q^2E$. When we discussed $\mathcal{U}_q(\mathfrak{sl}(2,k))$, we assumed q was an ℓ-th root of unity with $\ell \geq 3$ odd. When working with $\overline{\mathcal{U}}_q(\mathfrak{b}_+)$ is it more natural to make the substitution $q \mapsto q^2$ and this is permitted since q^2 is still an ℓ-th root of unity because $\ell \geq 3$ was assumed odd.

Recall that we showed $\mathcal{U}_q(\mathfrak{b}_+)$ was self-dual. With respect to the pairing (and $q \mapsto q^2$) we had

$$\langle E, E^m K^n\rangle = \delta_{m,1} \quad \text{and} \quad \langle K, E^m K^n\rangle = q^{2n}\delta_{m,0}.$$

To describe $D(\overline{\mathcal{U}}_q(\mathfrak{b}_+))$ we need to describe the multiplication and comultiplication on $(\overline{\mathcal{U}}_q(\mathfrak{b}_+)^{\mathrm{op}})^*$.[85] Also recall that the Hopf algebra structure on the dual is defined in terms of the pairing. So to deduce the multiplication and comultiplication of of $(\overline{\mathcal{U}}_q(\mathfrak{b}_+)^{\mathrm{op}})^*$ we need to investigate how the pairing for $\mathcal{U}_q(\mathfrak{b}_+)$ acts on $(\overline{\mathcal{U}}_q(\mathfrak{b}_+)^{\mathrm{op}})^*$. This will be significantly easier if we have a basis for $(\overline{\mathcal{U}}_q(\mathfrak{b}_+)^{\mathrm{op}})^*$, so we will describe such a basis first. Begin by defining functionals η and α on $\overline{\mathcal{U}}_q(\mathfrak{b}_+)^{\mathrm{op}}$ on a basis by

$$\langle \eta, E^m K^n\rangle := \delta_{m,1} \quad \text{and} \quad \langle \alpha, E^m K^n\rangle := q^{2n}\delta_{m,0}.$$[86]

It can be shown that these functionals linearly extend to all of $\overline{\mathcal{U}}_q(\mathfrak{b}_+)^{\mathrm{op}}$ and hence belong to $(\overline{\mathcal{U}}_q(\mathfrak{b}_+)^{\mathrm{op}})^*$. Moreover, we have the following theorem which we will not prove:

Theorem 20.1. $\{\eta^i\alpha^j\}_{0\leq i,j\leq \ell-1}$ is a basis for $(\overline{\mathcal{U}}_q(\mathfrak{b}_+)^{\mathrm{op}})^*$.

By induction and using properties of the pairing, its not too hard to show

$$\langle \eta^i, E^m K^n\rangle = [i]_{q^2}!\delta_{m,i} \quad \text{and} \quad \langle \alpha^j, E^m K^n\rangle = q^{2jn}\delta_{m,0}.$$

These relations together give a functional

$$\langle \eta^i\alpha^j, E^m K^n\rangle = [i]_{q^2}! q^{2j(i+n)}\delta_{m,i}$$

[85] We already know $(\overline{\mathcal{U}}_q(\mathfrak{b}_+)^{\mathrm{op}})^*$ is Hopf algebra so all we are doing is figuring out what the multiplication and comultiplication looks like.

[86] Since the paring is the evaluation map, this is equivalent to defining $\eta(E^m K^n) := \delta_{m,1}$ and $\alpha(E^m K^n) := q^{2n}\delta_{m,0}$.

which can be extended to a linear functional on all of $\overline{\mathcal{U}}_q(\mathfrak{b}_+)^{\text{op}}$. This turns out to be the multiplication on $(\overline{\mathcal{U}}_q(\mathfrak{b}_+)^{\text{op}})^*$. Using this multiplication, we can check $\alpha^\ell = 1$, $\eta^\ell = 0$, and $\alpha\eta\alpha^{-1} = q^{-2}\eta$. Similarly, we can check that

$$\langle (1 \otimes \eta) + (\eta \otimes \alpha), E^i K^j \otimes E^m K^n \rangle = \delta_{m,1} + \delta_{i,1} q^{2n} \delta_{m,0}$$

extends to a linear functional on $\overline{\mathcal{U}}_q(\mathfrak{b}_+)^{\text{op}}$ and describes the comultiplication on η, while

$$\langle \alpha \otimes \alpha, E^i K^j \otimes E^m K^n \rangle = q^{2j} \delta_{i,0} q^{2n} \delta_{m,0}$$

extends to a linear functional on $\overline{\mathcal{U}}_q(\mathfrak{b}_+)^{\text{op}}$ and describes the comultiplication on α. We can also check that the antipode acts on basis elements by

$$S(\alpha) = \alpha^{\ell-1} \quad \text{and} \quad S(\eta) = -\eta\alpha^{\ell-1}.$$

This finishes our discussion of describing the multiplication and comultiplication of $(\overline{\mathcal{U}}_q(\mathfrak{b}_+)^{\text{op}})^*$.

The quantum double $D(\overline{\mathcal{U}}_q(\mathfrak{b}_+))$ of $\overline{\mathcal{U}}_q(\mathfrak{b}_+)$ has as a k-vector space basis $\{\eta^i\alpha^j \otimes E^m K^n\}_{0 \le i,j,m,n \le \ell-1}$. We would like to see how this Hopf algebra is resembles $\overline{\mathcal{U}}_q(\mathfrak{sl}(2,k))$. Consider the map

$$\chi : D(\overline{\mathcal{U}}_q(\mathfrak{b}_+)) \to \overline{\mathcal{U}}_q(\mathfrak{sl}(2,k))$$
$$\eta^i\alpha^j \otimes E^m K^n \mapsto \left(\frac{q-q^{-1}}{q^2}\right)^i q^{2(i+j)m - i(i-1)} F^i E^m K^{i+j+n}$$

where it is defined on generators by $1 \otimes K \mapsto K$, $1 \otimes E \mapsto E$, $\alpha \otimes 1 \mapsto K$, and $\eta \mapsto \frac{q-q^{-1}}{q^2} FK$. This immediately implies χ is surjective since we know $\{E^i F^j K^m\}_{0 \le i,j,m \le \ell-1}$ is a basis for $\overline{\mathcal{U}}_q(\mathfrak{sl}(2,k))$. It's not difficult to check that χ is a Hopf algebra morphism. This suffices to checking that there are relations in $D(\overline{\mathcal{U}}_q(\mathfrak{b}_+))$ which under χ map to relations defining $\overline{\mathcal{U}}_q(\mathfrak{sl}(2,k))$. This tells us $D(\overline{\mathcal{U}}_q(\mathfrak{b}_+))$ resembles $\overline{\mathcal{U}}_q(\mathfrak{sl}(2,k))$ since to every relation in $\overline{\mathcal{U}}_q(\mathfrak{sl}(2,k))$ there is a corresponding relation in $D(\overline{\mathcal{U}}_q(\mathfrak{b}_+))$ mapping to it under χ. In fact, χ is even better. If R_D is the universal R-element of $D(\overline{\mathcal{U}}_q(\mathfrak{b}_+))$, then the universal R-element R of $\overline{\mathcal{U}}_q(\mathfrak{sl}(2,k))$ is given by $(\chi \otimes \chi)(R_D)$. This is immediately true since χ is a surjective morphism of Hopf algebras and so all the properties R needs to satisfy are preserved by χ. Explicitly,

$$R = \sum_{i \in I} \chi(1 \otimes e_i) \otimes \chi(e^i \otimes 1)$$

which provides an answer for how we naturally construct the universal R-element of $\overline{\mathcal{U}}_q(\mathfrak{sl}(2,k))$. In particular, we have finished answering our five questions.

We have $\{E^m K^n\}_{0\le m,n\le \ell-1}$ as a basis for $\overline{\mathcal{U}}_q(\mathfrak{b}_+)$ and $\{\eta^i\alpha^j\}_{0\le i,j\le \ell-1}$ as a basis for $(\overline{\mathcal{U}}_q(\mathfrak{b}_+)^{\mathrm{op}})^*$, but they are not dual to each other. So, we would like to find the dual basis to $\{E^m K^n\}_{0\le m,n\le \ell-1}$ described in terms of the basis $\{\eta^i\alpha^j\}_{0\le i,j\le \ell-1}$. Let $\{\beta^{i,j}\}_{0\le i,j\le \ell-1}$ be the dual basis to $\{E^m K^n\}_{0\le m,n\le \ell-1}$. Then

$$\beta^{i,j} = \sum_{0\le r,s\le \ell-1} \mu^{i,j}_{r,s}\eta^r\alpha^s$$

for some structure constants $\mu^{i,j}_{r,s} \in k$. So we are reduced to computing the structure constants $\mu^{i,j}_{r,s}$. We can use the fact that

$$\langle \eta^i\alpha^j, E^m K^n\rangle = [i]_{q^2}! q^{2j(i+n)}\delta_{m,i}$$

is zero unless $i = m$, to show $\mu^{i,j}_{r,s} = 0$ unless $r = i$. So we are further reduced to computing the structure constants $\mu^{i,j}_{i,s}$. This can be done in two ways. We could solve a system of linear equations obtained by applying the pairing to $E^m K^n$ and both sides of the equation above. It turns out that this method involves inverting a certain Vandermonde matrix. Alternatively, we may express the universal R-element by

$$R = \sum_{0\le i,j,s\le \ell-1} \mu^{i,j}_{i,s}(\chi(1\otimes E^i K^j)\otimes \chi(\eta^i\alpha^s)),$$

and by re-indexing we may further write

$$R = \sum_{0\le i,j,k\le \ell-1} c_{i,j,k}(E^k K^i \otimes F^k K^j).$$

Therefore it suffices to compute the $c_{i,j,k}$. Recall that $\tau\circ\Delta = R\Delta R^{-1}$ as functions. If we evaluate both sides at E and F separately we get recurrence relations in $c_{i,j,k}$. Using these relations it suffices to compute $c_{0,0,0}$ to compute all the $c_{i,j,k}$. Equating coefficients of $K^i\otimes K^i\otimes K^j$ in $(\Delta\otimes \mathrm{id})(R) = R_{(1,3)}R_{(2,3)}$ shows $c_{0,0,0} = 1/\ell$. Using the recurrence relations again we can show

$$c_{i,j,k} = \frac{1}{\ell}\frac{(q-q^{-1})^k}{[k]_q!}q^{k(k-1)/2+2k(i-j)-2ij}.$$

This identity can be used to show that R can be expressed as

$$R = R_K e_{q^{-2}}^{(q-q^{-1})(E\otimes F)}$$

which was our original expression for R.

We can now explain how $\overline{\mathcal{U}}_q(\mathfrak{sl}(2,k))$ produces a supply of quantum Yang-Baxter equations. Recall that $L(n,\epsilon)$ with $0 \le n \le \ell-1$ and $\epsilon = \pm 1$ are simple left $\overline{\mathcal{U}}_q(\mathfrak{sl}(2,k))$-modules with basis $\{m_i\}_{0\le i\le \ell-1}$. Then $\{m_i \otimes m_j\}_{0\le i,j\le \ell-1}$ is a basis for $L(n,\epsilon)\otimes L(n,\epsilon)$ and R is realized as a matrix by applying the representation defined on a the basis by $m_i \otimes m_j \mapsto R(m_i \otimes m_j)$. This matrix will satisfy a quantum Yang-Baxter equation.

Example 20.1 (A quantum Yang-Baxter equation solution from a quantum group)**.** Take $n = 1$ and $\epsilon = 1$. Then $L(n,\epsilon)$ is a 2-dimensional module with basis $\{m_0, m_1\}$. Then

$$R(m_0 \otimes m_0) = \sum_{0\le i,j,k\le \ell-1} c_{i,j,k}(E^k K^i m_0 \otimes F^k K^j m_0).$$

Recall $Em_0 = 0$ so the summands are zero unless $k = 0$. This implies $R(m_0 \otimes m_0)$ is a q-multiple of $(m_0 \otimes m_0)$. The computations are similar for the other basis elements of $L(n,\epsilon)\otimes L(n,\epsilon)$. In the end, up to scalar multiples, R as a matrix is of the form

$$\begin{pmatrix} q\lambda & 0 & 0 & 0 \\ 0 & \lambda & (q-q^{-1}) & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & q\lambda \end{pmatrix}$$

where $\lambda = q^{(\ell-1)/2}$.

This completes our discussion of how quantum groups connect to quantum Yang-Baxter equations!

21 Introduction to Semisimple Lie Algebras and Root Systems

In the following, all vector spaces are assumed finite dimensional. We'd like to discuss the representation theory of $\mathcal{U}_q(\mathfrak{g})$ for semisimple Lie algebras $\mathfrak{g}$. To do this we first need to introduce semisimple Lie algebras. For the moment, let k be a field of characteristic 0.

A Lie algebra $\mathfrak{g}$ over k is a k-vector space with a k-linear map called the bracket operation (or simply the bracket) $[\cdot,\cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$[87] which is anticommutative and satisfies the Jacobi identity:

$$[[x,y],z] + [[y,z],x] + [[z,x],y] = 0.$$

A homomorphism of Lie algebras is a k-linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ of Lie algebras such that $\phi([x,y]) = [\phi(x), \phi(y)]$. We say a subspace $\mathfrak{h}$ of $\mathfrak{g}$ is a Lie subalgebra if $\mathfrak{h}$ is closed under the bracket operation. We say $\mathfrak{i}$ is an ideal of $\mathfrak{g}$ if it is a subspace and $[x,y] \in \mathfrak{i}$ for all $x \in \mathfrak{i}$ and $y \in \mathfrak{g}$.[88] In particular, $\mathfrak{i}$ is a Lie subalgebra. The derived Lie algebra of $\mathfrak{g}$, denoted $[\mathfrak{g},\mathfrak{g}]$, is subspace of $\mathfrak{g}$ consisting of all linear combinations of Lie bracket pairs of elements of $\mathfrak{g}$. Clearly it is an ideal and hence a Lie subalgebra. The derived series of $\mathfrak{g}$ is then the sequence of Lie subalgebras

$$\mathfrak{g} \supseteq [\mathfrak{g},\mathfrak{g}] \supseteq [[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]] \supseteq \cdots .$$

If the series terminates at the zero Lie subalgebra 0 then $\mathfrak{g}$ is said to be solvable.[89] The radial of $\mathfrak{g}$, denoted $\mathrm{rad}(\mathfrak{g})$, is the unique maximal solvable ideal. The existence and uniqueness of the radical is not difficult to show. If $\mathfrak{g}$ has no proper ideals and $[\mathfrak{g},\mathfrak{g}] \neq 0$, we say $\mathfrak{g}$ is simple. More generally, if $\mathrm{rad}(\mathfrak{g}) = 0$ then we say $\mathfrak{g}$ is semisimple. So how are semisimple Lie algebras more general? This is seen in the following theorem:

Theorem 21.1. If $\mathfrak{g}$ is a semisimple Lie algebra, then $\mathfrak{g}$ is a direct sum of its simple ideals.

We would like to classify semisimple Lie algebras when k is algebraically closed. It suffices to classify simple Lie algebras. In the following, let k be an algebraically closed field of characteristic 0. If V is any vector space, then we can make $\mathrm{End}(V)$

[87] Think of the bracket as the multiplication in $\mathfrak{g}$.
[88] This is analogous to the ideal of a ring.
[89] This is analogous to the derived series of commutator subgroups in group theory.

into a Lie algebra by declaring that the bracket operation is the commutator. In other words, if $g, h \in \operatorname{End}(V)$ then $[g,h] = (g \circ h) - (h \circ g)$. A Lie algebra representation of $\mathfrak{g}$ is a Lie algebra homomorphism $\rho : \mathfrak{g} \to \operatorname{End}(V)$. In other words, the bracket of $\mathfrak{g}$ becomes the commutator in the homomorphic image. For any $x \in \mathfrak{g}$, there is a Lie algebra endomorphism

$$\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g} \qquad y \mapsto [x,y].$$

This induces a Lie algebra representation

$$\mathrm{Ad} : \mathfrak{g} \to \operatorname{End}(\mathfrak{g}) \qquad x \mapsto \mathrm{ad}_x$$

called the adjoint representation. It is extremely important throughout the entire theory.

Suppose $\mathfrak{g}$ is a semisimple Lie algebra and there exists an $x \in \mathfrak{g}$ such that $x^n \neq 0$ (the multiplication here is the bracket operation) for all $n \geq 1$.[90] We say that a Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is toral if $\mathrm{Ad}(\mathfrak{h})$ consists of semisimple elements.[91] Let $\mathfrak{t}$ denote a maximal toral subalgebra (i.e., not contained in a larger toral subalgebra). It can be shown $\mathfrak{t} \neq 0$ and $\mathfrak{t}$ is unique up to conjugation. Since elements of $\mathfrak{t}$ commute under Ad (recall semisimple is equivalent to diagonalizable in the finite dimensional setting), $\{\mathrm{ad}_t\}_{t\in\mathfrak{t}}$ is a commuting set of endomorphisms so it is simultaneously diagonalizable by the spectral theorem. Therefore there are eigenspaces $\mathfrak{g}_\alpha$ defined by

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [t,x] = \alpha(t)x\}$$

where $\alpha \in \mathfrak{t}^*$. Let Φ be the set of nonzero α such that $\mathfrak{g}_\alpha$ is nonzero. It follows that $\mathfrak{g}$ has a Cartan (or root space) decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$$

where we call Φ the root system of $\mathfrak{g}$. It turns out that it suffices to classify all root systems (but this is not an obvious fact) instead of classifying all semisimple Lie algebras.

Example 21.1 (Decomposition of $\mathfrak{sl}(2,k)$)**.** We define the Lie algebra $\mathfrak{sl}(2,k)$ by

$$\mathfrak{sl}(2,k) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in k, a+d = 0 \right\}$$

[90]This is a necessary technical condition.

[91]This makes sense since $\mathfrak{g}$ is finite dimensional so $\mathrm{Ad}(\mathfrak{h})$ is a space of matrices where the notion of semisimple makes sense.

where the bracket operation is the commutator. If we let

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

then $\mathfrak{sl}(2,k)$ has presentation

$$\langle E, F, H \mid [H,E] = 2E, [H,F] = -2F, [E,F] = H \rangle.$$

Observe that this generalizes our description of $\mathfrak{sl}(2,\mathbb{C})$. Moreover, it's not hard to check that $\mathfrak{t} = kH$ (the set of all k-multiples of H) is a maximal toral subalgebra. This should be suggestive however since H itself is diagonalizable, clearly E and F are not, and $\{E, F, H\}$ generate $\mathfrak{sl}(2,k)$. The relations $[H,E] = 2E$ and $[H,F] = -2F$ then imply that $\mathfrak{sl}(2,k)$ admits a decomposition

$$\mathfrak{sl}(2,k) = \mathfrak{t} \oplus \mathfrak{sl}(2,k)_\alpha \oplus \mathfrak{sl}(2,k)_{-\alpha}$$

where $E \in \mathfrak{sl}(2,k)_\alpha$ and $F \in \mathfrak{sl}(2,k)_{-\alpha}$.

We will take an axiomatic approach to root systems. We say that a subset Φ of a finite dimensional Euclidean vector space E with the standard Euclidean inner product $(\cdot,\cdot) : E \times E \to \mathbb{R}$ is a reduced root system if the following properties hold:

1. Φ is finite, spans E, and $0 \notin E$.

2. If $\alpha \in \Phi$ and $m\alpha \in E$ then $m = \pm 1$.

3. If $\alpha, \beta \in \Phi$, then the linear transformation defined by

$$r_\alpha(\beta) := \beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha$$

leaves Φ invariant.

4. If $\alpha, \beta \in \Phi$, then the inner product defined by

$$\langle \beta, \alpha \rangle := 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}$$

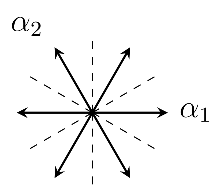
is in $\mathbb{Z}$.

The third property says that Φ is closed under the reflection of the hyperplane perpendicular to α (this reflection is r_α). The last property says that the projection of β onto the line through α is an integer or half-integer multiple of α. It is not obvious that the root system Φ of a semisimple Lie algebra $\mathfrak{g}$ satisfies the axioms of a root system just given, but it does.

If Φ is a root system it also has a basis. There exists a subset $\Delta \in \Phi$, called the set of simple roots, such that every root can be written as a unique $\mathbb{Z}$-linear combination of the simple roots. Moreover, all of the coefficients are either nonnegative or nonpositive. If the coefficient are nonnegative we say the root is positive and if the coefficients are nonpositive we say the root is negative. Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be the set of simple roots of Φ. Then we can form an $n \times n$ matrix $A = (a_{i,j})$ where we define $a_{i,j} := \langle \alpha_i, \alpha_j \rangle$. We call A the generalized Cartan matrix associated to Φ. It satisfies the following properties:

1. $a_{i,i} = 2$ for all i.
2. $a_{i,j} \leq 0$ for all $i \neq j$.
3. $a_{i,j} = 0$ implies $a_{j,i} = 0$.
4. $A = DS$ where D is an $n \times n$ diagonal matrix and S is an $n \times n$ symmetric matrix.

Example 21.2 (Root system for $\mathfrak{sl}(3, k)$)**.** It can be shown that the root system for $\mathfrak{sl}(3, k)$, called A_2, has a basis consisting of two simple roots (viewed in the plane) $\alpha_1 = (1, 0)$ and $\alpha_2 = (-1/2, \sqrt{3}/2)$. In particular, α_1 and α_2 make an angle of $2\pi/3$ between each other. The entire root system is the set $\{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2)\}$. Pictorially,



where the other solid arrows are the nonsimple roots and the dashed lines are the reflections of the hyperplanes perpendicular to the roots. The generalized Cartan matrix is

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

The amazing fact is that the entire theory can be run in reverse. That is, one can start with a generalized Cartan matrix and associate it a realization (we will not pursue this) which essentially contains both the root data and the Lie algebra data associated to the generalized Cartan matrix. In fact, generalized Cartan matrices and their realizations let us extend the theory to the infinite dimensional setting. See J. Humphreys's *Introduction to Lie algebras and representation theory* for a deep treatment of Lie algebras and root systems and see V. Kac's *Infinite Dimensional Lie Algebras* for the infinite dimensional setting.
Finally, we should mention that the reflections $\{r_{\alpha_i}, \dots, r_{\alpha_n}\}$ generate a group called the Weyl group W (or sometimes $W(\Phi)$) of Φ. This group is critical in the general theory, but we will only use it sparingly later.

22 A Treatment of $\mathcal{U}_q(\mathfrak{g})$ for Semisimple Lie Algebras $\mathfrak{g}$

The following mimics our discussion of the representation theory of $\mathcal{U}_q(\mathfrak{sl}(2,k))$. Let $\mathfrak{g}$ be a semisimple Lie algebra and let $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ be the set of simple roots. There is a presentation for $\mathcal{U}_q(\mathfrak{g})$ as

$$\mathcal{U}_q(\mathfrak{g}) = \langle K_i, K_i^{-1}, E_i, F_i \text{ for } 1 \le i \le r \mid R\rangle$$

where R stands for the set of relations

$$\begin{gathered} K_i K_i^{-1} = 1 = K_i^{-1} K_i, \quad K_i K_j = K_j K_i, \\ K_i E_j K_i^{-1} = q^{(\alpha_i,\alpha_j)} E_j, \quad K_i F_j K_i^{-1} = q^{-(\alpha_i,\alpha_j)} F_j, \\ E_i F_j - F_j E_i = \delta_{i,j} \tfrac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \\ \textstyle\sum_{m=0}^{1-a_{i,j}} (-1)^m \left[{1-a_{i,j} \atop m} \right]_{q_i} E_i^{1-a_{i,j}-m} E_j E_i^m = 0, \\ \textstyle\sum_{m=0}^{1-a_{i,j}} (-1)^m \left[{1-a_{i,j} \atop m} \right]_{q_i} F_i^{1-a_{i,j}-m} F_j F_i^m = 0. \end{gathered}$$

where $q_i = q^{(\alpha_i,\alpha_i)/2}$. Some authors call all of the relations above the Serre relations. We will only refer to the last two as the Serre relations. The algebra without the Serre relations, denoted $\tilde{\mathcal{U}}_q(\mathfrak{g})$, is sometimes useful. In general, there are other algebras we can consider. The $\mathbb{Z}$-span of Φ forms a lattice in Euclidean space. It is often denoted $\mathbb{Z}\Phi$ and called the root lattice. However, we can define a sort of dual lattice. We do this by forming a dual basis to Δ with respect to the pairing $\langle \cdot, \cdot \rangle$ (this is not hard to do). Denote this dual basis by $\{\omega_1, \ldots, \omega_r\}$. It is called the set of fundamental weights. It satisfies

$$\langle \omega_i, \alpha_j \rangle = \delta_{i,j}$$

It turns out that $\{\omega_1, \ldots, \omega_r\}$ is the basis of a root system and so there is an analogous notion of positive and negative weights (i.e., all coefficients are nonnegative or nonpositive). The $\mathbb{Z}$-linear space of the fundamental weights, denoted Λ, is called the weight lattice. One can identify Λ with $\mathrm{Hom}(\mathfrak{t}, \mathbb{C}^*)$. In general, one of $\mathbb{Z}\Phi$ or Λ is of finite index in the other one. If we let L be a lattice between these two lattices, then we can build a quantum group $\mathcal{U}_q(\mathfrak{g}, L)$ from $\mathfrak{g}$ with respect to L. In particular, we define $\mathcal{U}_q(\mathfrak{g}, L)$ by

$$\mathcal{U}_q(\mathfrak{g}, L) := \langle K_\mu, E_i, F_i \text{ for } \mu \in L \text{ and } 1 \le i \le r \mid R_L\rangle$$

where R_L stands for the set of relations

$$\begin{gathered} K_\lambda K_\mu = K_{\lambda+\mu}, \\ K_\lambda E_i K_\lambda^{-1} = q^{(\lambda,\alpha_i)} E_i, \quad K_\lambda F_i K_\lambda^{-1} = q^{(\lambda,\alpha_i)} F_i, \\ E_i F_j - F_j E_i = \delta_{i,j} \frac{K_{\alpha_i} - K_{\alpha_i}^{-1}}{q_i - q_i^{-1}}, \\ \sum_{m=0}^{1-a_{i,j}} (-1)^m \begin{bmatrix} 1-a_{i,j} \\ m \end{bmatrix}_{q_i} E_i^{1-a_{i,j}-m} E_j E_i^m = 0, \\ \sum_{m=0}^{1-a_{i,j}} (-1)^m \begin{bmatrix} 1-a_{i,j} \\ m \end{bmatrix}_{q_i} F_i^{1-a_{i,j}-m} F_j F_i^m = 0. \end{gathered}$$

So why is this generalization useful? Well, it turns out that there are instances where two different Lie groups give rise to the same semisimple Lie algebra. The prototypical example is $\mathrm{SL}(2,k)$ and $\mathrm{PGL}(2,k)$ both with Lie algebra $\mathfrak{sl}(2,k)$. However, one of these two Lie groups is more naturally associated with the root lattice and the other with the weight lattice. The above construction also us to distinguish these Lie groups by associating different quantum groups via lattices. We should mention that in the more verbose notation $\mathcal{U}_q(\mathfrak{g}, L)$ we will write $\mathcal{U}_q(\mathfrak{g})$ for $\mathcal{U}_q(\mathfrak{g}, \mathbb{Z}\Phi)$. We will still write $\mathcal{U}_q(\mathfrak{g})$ in this case.

From now on we will assume q is not a root of unity. We would like to describe the representation theory of $\mathcal{U}_q(\mathfrak{g})$ and its structure as a quantum group. To understand the representation theory, we will use the natural embeddings $\mathcal{U}_{q_i}(\mathfrak{sl}(2,k)) \to \mathcal{U}_q(\mathfrak{g})$ given by the obvious mapping on generators. Our end goal will be to classify the finite dimensional simple left $\mathcal{U}_q(\mathfrak{g})$-modules. It turns out that there are $2^{|\Delta|}$ finite dimensional simple left $\mathcal{U}_q(\mathfrak{g})$-modules for every finite dimensional simple left $\mathfrak{g}$-module. This is a generalization in the $\mathfrak{sl}(2,\mathfrak{k})$ setting where we had two finite dimensional simple left $\mathcal{U}_q(\mathfrak{sl}(2,k))$-modules $L(n,\epsilon)$ for every finite dimensional simple left $\mathfrak{sl}(2,k)$-module $L(n)$. To begin, we have a decomposition theorem:

Proposition 22.1. Let M be a finite dimensional simple left $\mathcal{U}_q(\mathfrak{g})$-module. Then M is the direct sum of its weight spaces $M_{\lambda,\sigma}$ defined by

$$M_{\lambda,\sigma} := \{m \in M \mid K_\mu m = \sigma(\mu) q^{(\lambda,\mu)} m \text{ for all } \mu \in \mathbb{Z}\Phi\}$$

where $\lambda \in \Lambda$ and $\sigma : \mathbb{Z}\Phi \to \{\pm 1\}$ is a homomorphism. Moreover, $E_i M_{\lambda,\sigma} \subseteq M_{\lambda+\alpha_i,\sigma}$, $F_i M_{\lambda,\sigma} \subseteq M_{\lambda-\alpha_i,\sigma}$, and E_i and F_i act as nilpotent operators on M.

Proof sketch. The statements regarding the actions of E_i and F_i are immediate from the relations. Now recall that for every $\alpha_i \in \Delta$, we have a natural embedding of $\mathcal{U}_{q_i}(\mathfrak{sl}(2,k))$ into $\mathcal{U}_q(\mathfrak{g})$. This implies that M is a left $\mathcal{U}_{q_i}(\mathfrak{sl}(2,k))$-module. Since E_i and F_i act as nilpotent operators on M as a left $\mathcal{U}_{q_i}(\mathfrak{sl}(2,k))$-module, they act as nilpotent operators on M as a left $\mathcal{U}_q(\mathfrak{g})$-module. Since M is diagonalizable with respect to the K_i, and K_i and K_j commute, M is simultaneously diagonalizable. From here it follows easily that every $m \in M$ belongs to some weight space $M_{\lambda,\sigma}$. This proves M decomposes into weight spaces. To show this decomposition is direct, argue by contradiction that m cannot belong to two distinct weight spaces. $\square$

Now write M as

$$M = \bigoplus_{\sigma} M^{\sigma}$$

where the sum is over all homomorphisms $\sigma : \mathbb{Z}\Phi \to \{\pm 1\}$ and where

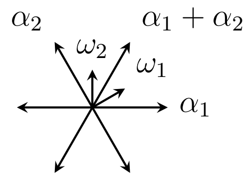
$$M^{\sigma} = \bigoplus_{\lambda \in \mathbb{Z}\Phi^{\vee}} M_{\lambda,\sigma}.$$

Each of the M^{σ} is a left $\mathcal{U}_q(\mathfrak{g})$-submodule necessarily finite dimensional. In fact, we can deduce more. If M and N are two finite dimensional left $\mathcal{U}_q(\mathfrak{g})$-modules with a homomorphism $\varphi : M \to N$, then the image of $\varphi(M_{\lambda,\sigma}) \subseteq N_{\lambda,\sigma}$. If φ is an isomorphism then the image is exactly $N_{\lambda,s}$. Therefore, φ preserves the decomposition of M above and we need only study the M^{σ}. So let M be of the form M^{σ} (so the above decomposition has only one term). If σ is the trivial homomorphism then we say M is of trivial type. If M is of trivial type and $\tau : \mathbb{Z}\Phi \to \{\pm 1\}$ is an arbitrary homomorphism then we may twist the action of $\mathcal{U}_q(\mathfrak{g})$ on M by the mapping $K_i \mapsto \tau(\alpha_i)K_i$, $K_i^{-1} \mapsto \tau(\alpha_i)K_i^{-1}$, $E_i \mapsto \tau(\alpha_i)E_i$, and $F_i \mapsto F_i$ on generators. This turns out to induce an automorphism of $\mathcal{U}_q(\mathfrak{g})$. Moreover, $M = M^{\sigma}$ (for σ trivial) becomes $M = M^{\tau}$ under this new action (the action by the image under the automorphism) and we say M is of τ-type. Since this twisting is an automorphism, it suffices to study only those modules of trivial type. So, we will make the convention from now on that we denote $M_{\lambda,\sigma}$ (where σ is trivial) by M_{λ}.

Before characterizing the representation theory of $\mathcal{U}_q(\mathfrak{g})$, lets review the representation theory of $\mathfrak{g}$. It turns out that finite dimensional left $\mathfrak{g}$-modules are characterized by highest weight vectors. Begin by introducing a partial ordering on weights given by $\lambda > \mu$ if $\lambda - \mu$ is a positive root.[92] We say a weight is highest if it is largest with respect to this partial ordering. It is not yet clear that there is a unique highest weight, but there is one in general. Let Λ^+ stand for the set of all positive weights. We will call them dominate weights from now on.

Example 22.1 (Highest weights for $\mathfrak{sl}(3, k)$)**.** Recall that the root system for $\mathfrak{sl}(3, k)$ is

[92]We really mean root here not weight.



where, unlike before, are making explicit the root $\alpha_1 + \alpha_2$ and suppressing the hyperplane reflections. Moreover, we display the fundamental weights ω_1 and ω_2 in the diagram. Clearly Λ^+ is the set of all nonnegative linear combinations of ω_1 and ω_2. Observe $\omega_1 + \omega_2$ gives $\alpha_1 + \alpha_2$. It turns out that $\omega_1 + \omega_2 = \alpha_1 + \alpha_2$ is the highest weight vector for $\mathfrak{sl}(3, k)$.

With this setup, we have an important theorem:

Theorem 22.1. If $\mathfrak{g}$ is a finite dimensional semisimple Lie algebra, its finite dimensional simple left $\mathfrak{g}$-modules are in bijective correspondence with Λ^+. The weight spaces of the modules are permuted by the Weyl group W and this preserves the dimension.

It it common practice, as in the $\mathfrak{sl}(2, k)$ setting, to denote the finite dimensional simple left $\mathfrak{g}$-module corresponding to $\lambda \in \Lambda^+$ by $L(\lambda)$.

This concludes our discussion of the representation theory of $\mathfrak{g}$, and begins our discussion of the representation theory of $\mathcal{U}_q(\mathfrak{g})$. If M is a finite dimensional simple left $\mathcal{U}_q(\mathfrak{g})$-module, then

$$M = \bigoplus_{\lambda \in \Lambda} M^\lambda$$

because we can assume each factor is of trivial type by the previous discussion. We also have a useful proposition:

Proposition 22.2. If M is a nonzero finite dimensional simple left $\mathcal{U}_q(\mathfrak{g})$-module, then

1. There exists $\lambda \in \Lambda$ with $v \in M_\lambda$ and $v \neq 0$ such that $E_i v = 0$.
2. For any such v, the corresponding λ is dominant and $F_j^{\langle \lambda, \alpha_j \rangle + 1} v = 0$.

Proof sketch. We have already concluded part (1). For the second part, use the $\mathcal{U}_{q_i}(\mathfrak{sl}(2, k))$ embedding. □

From this proposition, its not hard to show that there is a unique highest weight vector. Using the highest weights we can construct universal left $\mathcal{U}_q(\mathfrak{g})$-modules as in the $\mathfrak{sl}(2,k)$ setting. Form subalgebras $\mathcal{U}_q^+(\mathfrak{g}), \mathcal{U}_q^-(\mathfrak{g})$, and $\mathcal{U}_q^0(\mathfrak{g})$ as algebras generated by the E_i, F_i, and K_i and K_i^{-1} respectively. In particular, $\mathcal{U}_q^0(\mathfrak{g})$ is commutative. Then there is an isomorphism of algebras

$$\mathcal{U}_q^0(\mathfrak{g}) \otimes \mathcal{U}_q^0(\mathfrak{g}) \otimes \mathcal{U}_q^+(\mathfrak{g}) \cong \mathcal{U}_q(\mathfrak{g}).$$

We can define the universal left $\mathcal{U}_q(\mathfrak{g})$-modules $M(\lambda)$ by

$$M(\lambda) := \mathcal{U}_q(\mathfrak{g})/J_\lambda$$

where J_λ is the left ideal defined by

$$J_\lambda := \left(\sum_{i=1}^r \mathcal{U}_q(\mathfrak{g}) E_i + \sum_{i=1}^r \mathcal{U}_q(\mathfrak{g})(K_i - q^{(\lambda,\alpha_i)}) \right).$$

This module is universal as the following proposition shows:

Proposition 22.3. If M is a finite dimensional simple left $\mathcal{U}_q(\mathfrak{g})$-module with highest weight vector $v \in M_\lambda$, then there exists a a unique left $\mathcal{U}_q(\mathfrak{g})$-module homomorphism from $M(\lambda)$ to M by $[1] \mapsto v$.

There is quite a lot more work involved in the general theory, but one can shown a classification theorem for the finite dimensional $\mathcal{U}_q(\mathfrak{g})$-modules. We will take for granted that there is a unique maximal left $\mathcal{U}_q(\mathfrak{g})$-submodule of $M(\lambda)$. Call it W. Then we have the following classification theorem:

Theorem 22.2. Every finite dimensional simple left $\mathcal{U}_q(\mathfrak{g})$-module of trivial type is of the form $M(\lambda)/W$ for λ a dominant weight where W is the unique maximal left $\mathcal{U}_q(\mathfrak{g})$-submodule of $M(\lambda)$.

There is another theorem which tells us the following:

Theorem 22.3. Let $L(\lambda)$ be a finite dimensional simple left $\mathfrak{g}$-module or a finite dimensional simple left $\mathcal{U}_q(\mathfrak{g})$-module.[93] Then the dimension of the weight space $L(\lambda)_\mu$ for $\mu \in \Lambda$ is the same as that for the corresponding module.

There are several ways to compute this dimension. One way is the Kostant multiplicity formula:

$$\dim(K(\lambda)_\mu) = \sum_{\sigma \in W} (-1)^{\ell(w)} p(\mu + \delta - \sigma(\lambda + \delta)).$$

[93] The classification theorems imply that is is valid to write write the module as $L(\lambda)$.

We need to explain some of this notation. We define δ by

$$\delta := \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$$

with Φ^+ denoting the set of positive roots. We call p the Kostant partition function. In particular, $p(\lambda)$ is the number of ways of writing $-\lambda$ as a linear combination of positive roots. Also $\ell(w)$ is the length of the element $w \in W$. In other words, $\ell(w)$ is the minimum number of generators needed to write w (this can be shown to be well-defined).

We conclude by mentioning the Hopf algebra structure of $\mathcal{U}_q(\mathfrak{g})$. We define the Hopf algebra structure of $\mathcal{U}_q(\mathfrak{g})$ by demanding that the natural embeddings of $\mathcal{U}_{q_i}(\mathfrak{sl}(2,k))$ to be Hopf algebra morphism. This suffices to characterize the Hopf algebra structure of $\mathcal{U}_q(\mathfrak{g})$ completely. Indeed, just view the Hopf algebra structures for the $\mathcal{U}_{q_i}(\mathfrak{sl}(2,k))$ under the natural embedding (assuming they're now Hopf algebra morphisms) and this will naturally give rise to definitions for the Hopf algebra structure of $\mathcal{U}_q(\mathfrak{g})$. Then just check that those definitions make $\mathcal{U}_q(\mathfrak{g})$ into a Hopf algebra.

23 Crystal Bases

In the following we will assume q is not a root of unity. The goal of crystal bases is to find bases of left $\mathcal{U}_q(\mathfrak{g})$-modules for semisimple Lie algebras $\mathfrak{g}$ that behave well with respect to raising and lowering operators on weight spaces. Recall that for $\mathfrak{U}_q(\mathfrak{sl}(2,k))$-modules we've already seen that the E_i and F_i act as raising and lowering operators. In general, for any semisimple Lie group $\mathfrak{g}$ it is not the case that all the E_i and F_i will act in this way on $\mathcal{U}_q(\mathfrak{g})$-modules. But it is possible, in a lose sense, by talking the limit as $q \to 0$.

Let M be a finite dimensional left $\mathcal{U}_q(\mathfrak{g})$-module viewed as a left $\mathcal{U}_{q_i}(\mathfrak{sl}(2,k))$-module written as

$$M = L_1 \oplus \cdots \oplus L_t$$

where the $L_j = L(n_j)$ are simple left $\mathcal{U}_{q_i}(\mathfrak{sl}(2,k))$-modules with $n_j \geq 0$. Let m_j be the highest weight vector for L_j. Then M has a basis $\{F_i^k m_j\}_{\substack{k\geq 0\\ i\leq j\leq t}}$. We will normalize F_i by defining

$$F_i^{(k)} := F_i^k/[k]_{q_i}!.$$

Given any $x \in M$, $K_i x = q_i^\ell x$ for some ℓ. Then we can write

$$x = \sum_{k\geq 0} \sum_{j:n_j-2k=\ell} a_j F_i^{(k)} m_j = \sum_{k\geq 0} F_i^{(k)} x_k$$

where $j : n_j - 2k = \ell$ means to sum over all j such that $n_j - 2k = \ell$ and we define

$$x_k := \sum_{j:n_j-2k=\ell} a_j m_j.$$

Observer that the x_k are uniquely determined by their weight spaces and that $E_i x_k = 0$. For $x \in M$ we define raising and lowering operators respectively by

$$\tilde{E}_i x := \sum_{k\geq 0} E_i^{(k-1)} x_k \quad \text{and} \quad \tilde{F}_i x := \sum_{k\geq 0} F_i^{(k+1)} x_k.$$

The upshot here is that with respect to a well-chosen basis, these operators will give rise to a crystal bases which we will now introduce.

Set $k = \mathbb{Q}(q)$, the fraction field of $\mathbb{Q}[q]$ where we only assume q is transcendental over $\mathbb{Q}$. Inside k is a local ring A defined by

$$A = \left\{ \frac{f}{g} \mid f, g \in \mathbb{Q}[q] \text{ with } g(0) \neq 0 \right\}.$$

It has (q) as a maximal ideal and $A/(q) \cong \mathbb{Q}$ by evaluation at 0. Given a finite dimensional left $\mathcal{U}_q(\mathfrak{g})$-module M, we say that a left A-submodule $\mathcal{M}$ is an admissible lattice for M if the following properties are satisfied:

1. $\mathcal{M}$ is finitely generated over A and generates M as a vector space over k.
2. $\mathcal{M} = \bigoplus_{\lambda \in \Lambda} \mathcal{M} \cap M_\lambda$.
3. The raising and lowering operators satisfy $\tilde{F}_i \mathcal{M} \subseteq \mathcal{M}$ and $\tilde{E}_i \mathcal{M} \subseteq \mathcal{M}$.

In particular, $\mathcal{M}$ is a free left A-module of finite rank such that $\mathcal{M} \otimes_A k \cong M$ given by $m \otimes c \to cm$. It can be shown that admissible lattices respect isomorphisms and direct sums. So we only need to describe admissible lattices when the module is simple. We now define a crystal basis. If M is a finite dimensional left $\mathcal{U}_q(\mathfrak{g})$-module then a crystal basis is a pair $(\mathcal{M}, \mathcal{B})$ where $\mathcal{M}$ is an admissible lattice for M and $\mathcal{B}$ is a basis for the $\mathbb{Q}$-vector space $\mathcal{M}/q\mathcal{M}$ such that the following properties are satisfied:

1. $\mathcal{B} = \bigcup_{\lambda \in \Lambda} (\mathcal{B} \cap \mathcal{M}_\lambda / q\mathcal{M})$.
2. $\tilde{F}_i \mathcal{B} = \mathcal{B} \cup \{0\}$ and $\tilde{E}_i \mathcal{B} = \mathcal{B} \cup \{0\}$.
3. If $b, b' \in \mathcal{B}$ then $b = \tilde{E}_i b'$ if and only if $b' = \tilde{F}_i b$.

It can be shown that crystal bases respect isomorphisms and direct sums so we only need to describe them when the module is simple.

Let's describe how to make a crystal basis for the simple modules $L(\lambda)$. Start by choosing a highest weight vector $v_\lambda \in L(\lambda)_\lambda$ and let $\mathcal{L}(\lambda)$ be the left A-submodule of $L(\lambda)$ spanned by all possible terms of the form

$$\tilde{F}_{i_1} \cdots \tilde{F}_{i_s} v_\lambda$$

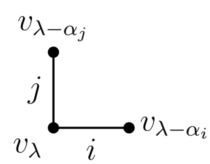
for $s \geq 0$ and $1 \leq i_1, \ldots, i_s \leq r$. Then show $\mathcal{L}(\lambda)$ is an admissible lattice for $L(\lambda)$. Set $\mathcal{B}(\lambda)$ to be the set of all nonzero terms of the form

$$\tilde{F}_{i_1} \cdots \tilde{F}_{i_s} \overline{v_\lambda}$$

for $s \geq 0$, $1 \leq i_1, \ldots, i_s \leq r$, and where $\overline{v_\lambda}$ is the image of v_λ in $\mathcal{L}(\lambda)/q\mathcal{L}$ under the natural projection. Then show $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$ is a crystal basis for $L(\lambda)$.

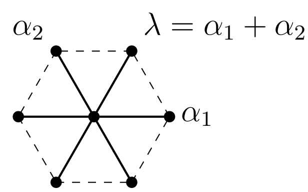
24 Crystal Graphs

Crystal bases have a beautiful model. The elements $b \in \mathcal{B}$ are of the form $b = \tilde{F}_{i_1} \cdots \tilde{F}_{i_s} \overline{v_\lambda}$. These can be modeled with a graph structure called a crystal graph as follows: there is a vertex in the graph for each $b \in \mathcal{B}$ and there is an edge with color i between b and b' if $\tilde{F}_i b = b'$. Observe that we don't create edges if $\tilde{F}_i b = 0$. So a subgraph of the crystal graph may looks like

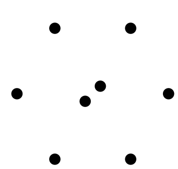


Let's now describe an explicit crystal graph so that we go through the construction and describe some interesting observations along the way.

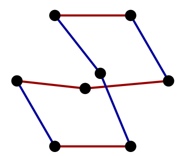
Example 24.1 (Crystal graph for $\mathfrak{sl}(3, k)$)**.** Recall that the root system for $\mathfrak{sl}(3, k)$, called A_2, had a basis (in the plane) consisting of two simple roots α_1 and α_2. We also know that $\lambda = \alpha_1 + \alpha_2$ is the highest weight vector and all the weights are in the W-orbit of λ and since λ is a root that's all the other roots. There is also the zero weight space at the origin (we say this even though zero is not a weight). So the weights lie on the dots contained in the diagram:



Moreover the weight space corresponding to λ is 1-dimensional and since the W-action preserves the dimension we know that all the weight spaces corresponding to the weights on the vertices of the octagon are 1-dimensional. It turns out that the weight space corresponding to the origin is 2-dimensional by the Konstant multiplicity formula. This implies that the crystal graph has 8 vertices and we draw them as



It's not difficult to determine edges between vertices that are the not the two at the center. For edges involving a vertices at the center it becomes a little more tricky because we need to distinguish between the two vertices at the center. It turns out that the crystal graph is



Recall that we've described a crystal basis for $L(\lambda)$. We now want to delve a little deeper into the strategy of how this was done. It's really a two step process. The first step is to prove crystal bases exist for small representations (i.e., low dimensional). The second is to show that if M and N are finite dimensional simple left $\mathcal{U}_q(\mathfrak{g})$-modules both with a crystal basis, then $M \otimes N$ has a natural crystal basis.

To the first step, this process suggests we need to choose a large enough collection of small representations such that if we take successive tensor products then all the finite dimensional simple left $\mathcal{U}_q(\mathfrak{g})$-modules will appear in these tensor products. What might this class be? The answer is to choose all representations whose highest weight is either minuscule or is the largest short root (we will explain this as well) in Φ. By a minuscule weight we mean $\lambda \neq 0$ and $\langle \lambda, \alpha \rangle \in \{\pm 1, 0\}$ for all $\alpha \in \Phi$. In other words, λ is a non-zero minimal dominant weight. These representations are particularly nice in that all the weights occur in the W-orbit of the highest weight and therefore all weight spaces are 1-dimensional. This is the situation for $\mathfrak{sl}(3,k)$ that we just saw. By largest short root, we mean that the weight in largest with respect to the partial ordering and is short in that it is of least length as a vector.[94] For $\mathfrak{sl}(3,k)$, all roots have the same length. By the previous comment these are essentially the only representations we need to study explicitly.

To the second step, we want to see if we can construct a crystal basis on the tensor product of two finite dimensional left $\mathcal{U}_q(\mathfrak{g})$-modules M and N if both M and N have a crystal basis. There is something subtle that needs to be done here. We need to change the comultiplication on $\mathcal{U}_q(\mathfrak{g})$ by twisting via the automorphism defined on generators by swapping the E_i and F_i generators and acting trivially on the other generators. This preserves our previous results and gives rise to isomorphic modules. So why do we need to do this? It's because the tensor product of admissible lattices is not admissible without this twisting action. We can now state an amazing theorem:

[94]In a reduced root system, it's a general fact that there are at most two lengths for roots.

Theorem 24.1. If L and M are finite dimensional simple left $\mathcal{U}_q(\mathfrak{g})$-modules with crystal bases $(\mathcal{L}, \mathcal{B}_1)$ and $(\mathcal{M}, \mathcal{B}_1)$ respectively, then $(\mathcal{L} \otimes \mathcal{M}, \mathcal{B}_1 \otimes \mathcal{B}_2)$ is a crystal basis for $L \otimes M$. Moreover, the operators $\tilde{F}_i$ and $\tilde{E}_i$ act on $\mathcal{B}_1 \otimes \mathcal{B}_2$ by

$$\tilde{F}_i(b_1 \otimes b_2) = \begin{cases} \tilde{F}_i b_1 \otimes b_2 & \text{if } f_i(b_1) > e_i(b_2) \\ b_1 \otimes \tilde{F}_i b_2 & \text{if } f_i(b_1) \leq e_i(b_2) \end{cases}$$

and

$$\tilde{E}_i(b_1 \otimes b_2) = \begin{cases} \tilde{E}_i b_1 \otimes b_2 & \text{if } e_i(b_1) > f_i(b_2) \\ b_1 \otimes \tilde{E}_i b_2 & \text{if } e_i(b_1) \leq f_i(b_2) \end{cases}$$

where $f_i(b) = \max\{r \mid \tilde{F}_i^r b \neq 0\}$ and $e_i(b) = \max\{r \mid \tilde{E}_i^r b \neq 0\}$.

There are two important consequences of this theorem. The first is that it's easy to give this algorithm to a computer, and so it becomes very easy to compute complicated crystal graphs. The second is that the connected components in a crystal graph of the tensor product are precisely the simple summands in product.

Appendices

A Hopf Algebras and Groups

We are going to show that Hopf algebras and groups are “the same” structure. Recall that a group G is a set with an associative binary operation $m : G \otimes G \to G$, an inverse map $S : G \to G$, and an identity element $\eta : * \to G$ where $*$ denotes the singleton set.[95] We would like to redefine the axiom for the inverse map $S : G \to G$. To do this, we need to utilize two natural maps that we get for free. The first is the obvious map $\epsilon : G \to *$. The second is the diagonal map

$$\Delta : G \to G \otimes G \qquad g \mapsto g \otimes g.$$

In fact, if we replace the field k in our discussion of algebras, coalgebras, and bialgebras with the singleton set $*$, then all the commutative diagrams for a bialgebra are satisfied by G and we encourage the interested reader to verify this. Notice we have not yet used the inverse map $S : G \to G$. We now (re)define a group G to be a bialgebra over $*$ (as discussed above) with a map $S : G \to G$ such that

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \epsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta.$$

In other words, we are asking that G satisfies the axiom for a Hopf algebra over $*$. In terms of elements of G, we can express the above identity as

$$S(g)g = 1 = gS(g).$$

In other words, $S(g)$ as the inverse element of g and $S : G \to G$ is the inverse map! So, we see that groups are just Hopf algebra over $*$ or Hopf algebras are just groups over an arbitrary field k. If we adopt $*$ as a field, then we can say groups instead of Hopf algebras!

This should shed some light on why we call a quasitriangular Hopf algebra a quantum group. The “quantum” in quantum group expresses the relaxation the cocommutativity (as we have said previously) and “group” in quantum group really just means Hopf algebra.

[95] We are being suggestive when we label the multiplication, inverse, and unit by m, s, and η respectively.