REU Notes

Henry Twiss

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Contents

1 Lattice Models

1.1 Theory

There is a natural action of S_n on the ring of polynomials in $x_1, \dots, x_n$ by permuting the variables. The symmetric functions in $x_1, \dots, x_n$ are the elements of $\mathbb{Z}[x_1, \dots, x_n]$ which are invariant under this action. Sometimes we denote $x_1, \dots, x_n$ by $\underline{x}$.

Example 1.1. Let $n = 3$. Then

$$\begin{aligned} e_2(x_1, x_2, x_3) &= x_1x_2 + x_2x_3 + x_1x_3 \\ h_2(x_1, x_2, x_3) &= x_1^2 + x_1x_2 + x_2^2 + x_2x_3 + x_3^2 + x_1x_3 \\ p_4(x_1, x_2, x_3) &= x_1^4 + x_2^4 + x_3^4 \\ m_{(2,1,1)}(x_1, x_2, x_3) &= x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2 \end{aligned}$$

are all symmetric functions. The first is an example of an elementary symmetric function, the second a power symmetric function, the third a homogeneous symmetric function, and the latter is a monomial symmetric function. Notice that the monomial symmetric functions are dependent on a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$ with $\lambda_i \geq 0$.

If we are given a partition λ, then we can define $p_\lambda(\underline{x})$ and $h_\lambda(\underline{x})$ by

$$p_\lambda := p_{\lambda_1}(\underline{x}) \cdots p_{\lambda_n}(\underline{x}) \quad \text{and} \quad h_\lambda := h_{\lambda_1}(\underline{x}) \cdots h_{\lambda_n}(\underline{x}).$$

Let S_n denote the symmetric group on n letters. Define $\Lambda_n := \mathbb{Z}[x_1, \dots, x_n]^{S_n}$; the symmetric functions. We have the following facts:

$$\Lambda_n \cong \mathbb{Z}[e_1, \dots, e_n] \quad \text{and} \quad \Lambda_n = \oplus_{k \geq 0} \Lambda_n^k$$

where Λ_n^k denotes the homogeneous symmetric functions of degree k. Moreover $\{m_\lambda\}_{\substack{\ell(l) \leq n \\ |\lambda| = k}}$ is a basis for Λ_n where $\ell(\lambda)$ is the number of parts of λ and $|\lambda|$ is the sum of the parts. There is a natural scalar product $\langle \cdot, \cdot \rangle$ (arising from the theory of Hopf algebras) on Λ_n defined by

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda,\mu}.$$

It turns out to be a symmetric inner product. We call it the Hall inner product. Naturally, one may ask if there is an orthonormal basis with respect to this inner product. The answer is yes and elements of the basis are called Schur polynomials. We will introduce them by giving two equivalent definitions of Schur polynomials:

1. We define the Schur polynomial $s_\lambda(x_1, \ldots, x_n)$ by

$$s_\lambda(x_1, \ldots, x_n) := \frac{A_{\lambda+\rho}}{A_\rho}$$

   where

$$A_\mu = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\sigma(\underline{x}^\mu)$$

   where $\underline{x}^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}$ and $\rho = (n-1, n-2, \ldots, 1, 0)$ is introduced to give $\lambda + \rho$ distinct parts. In particular notice that $A_\mu = 0$ if $\mu_i = \mu_{i+1}$ for some i.

2. Given a partition λ form the Young diagram with shape λ and fill it will the alphabet $\{1, \ldots, n\}$ to make a semistandard Young tableau. We write $\mathrm{SSYT}(\lambda)$ for the set of all semistandard Young tableau with shape λ. Then we define the Schur polynomial $s_\lambda(x_1, \ldots, x_n)$ by

$$s_\lambda(x_1, \ldots, x_n) := \sum_{T \in \mathrm{SSYT}(\lambda)} \underline{x}^{\mathrm{wt}(T)}$$

   where $\mathrm{wt}(T)$ is the weight of the tableau T.

There is also a third definition of Schur polynomials arising from statistical mechanics. For a deep treatment of statistical mechanics, the quantum Yang-Baxter equation, ice models, and more see chapters 2-7 (in particular chapters 2 and 7) of the author's Quantum Groups Booklet at https://themodularperspective.com/graduate-text-notes/.

1.2 Exercises

1.

(a) Compute $s_{(5,4,1)}$ using both definitions.

(b) Show A_μ can be expressed as the determinant of a matrix.

(c) Evaluate A_ρ as an explicit product in the x_is.

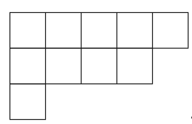
Proof.

(a) Using the first definition, we need to compute $A_{(7,5,1)}/A_{(2,1,0)}$. We have

$$A_{(7,5,1)} = \sum_{\sigma\in S_3} \operatorname{sgn}(\sigma)\sigma(x_1^7x_2^5x_3) = x_1^7x_2^5x_3 - x_2^7x_1^5x_3 - x_3^7x_2^5x_1 - x_1^7x_3^5x_2 + x_2^7x_3^5x_1 + x_3^7x_1^5x_2$$
$$A_{(2,1,0)} = \sum_{\sigma\in S_3} \operatorname{sgn}(\sigma)\sigma(x_1^2x_2) = x_1^2x_2 - x_2^2x_1 - x_3^2x_2 - x_1^2x_3 + x_2^2x_3 + x_3^2x_1.$$

Observe $A_{(2,1,0)} = \Delta(x_1, x_2, x_3)$. *Aprior* it is not clear that the ratio is a polynomial, but this can be checked quickly using compute software. In particular, we will notice that each monomial has coefficient 1 and the exponent vector is some partition of 10 into 3 parts with all parts at least 1.

Using the second definition, our Young diagram has shape



.

We fill the diagram with the alphabet $\{1, 2, 3\}$. All semistandard tableau with shape $(5, 4, 1)$ take one of the following forms

$$\begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&-\\\hline 2&2&2&2\\\cline{1-4} 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|c|c|}\hline 1&1&1&-&-\\\hline 2&2&2&3\\\cline{1-4} 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|c|c|}\hline 1&1&-&-&-\\\hline 2&2&3&3\\\cline{1-4} 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|c|c|}\hline 1&-&-&-&-\\\hline 2&3&3&3\\\cline{1-4} 3\\\cline{1-1}\end{array}.$$

From these partial tableau, notice that each monomial in the sum

$$s_{(5,4,1)} = \sum_{T\in \mathrm{SSYT}(\lambda)} \underline{x}^{\mathrm{wt}(T)}$$

has exponent weight vector $\mathrm{wt}(T)$ such that $|\mathrm{wt}(T)| = 10$ and the entries are some partition of 10 into 3 parts with each part ≥ 1. It is now clear that the two definitions agree.

(b) A_μ is the determinant of

$$\begin{pmatrix} x_1^{\mu_1} & x_1^{\mu_1} & \cdots & 1 \\ x_2^{\mu_2} & x_2^{\mu_2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{\mu_n} & x_n^{\mu_n} & \cdots & 1 \end{pmatrix}.$$

Indeed, by the Leibniz determinant formula, each monomial in the determinant has a permutation of μ as an exponent vector with sign corresponding to the sign of the permutation. This is the definition of A_μ.

(c) Specializing (b) to when $\mu = \rho$, this is easily seen to be the Vandermonde determinant. In other words,

$$A_\rho = \prod_{1 \le i < j \le n} (x_i - x_j).$$

□

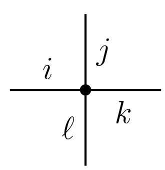
2. Determine the simple expression for $(*)$ in

$$P_\lambda(\underline{x}) = (*) s_\lambda(\underline{x})$$

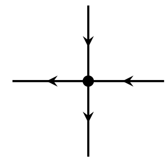
and prove the identity using the second definition of Schur functions.

3. Find a solution to the Yang-Baxter equation for the Boltzmann weights.

Proof. We could of course proceed in a brute force approach by computing all possible boundary configurations for the YBE and deduce what the weights of the cross configuration are. We are going to be a little more cleaver and instead do all these computations at once after a little bit of abstract setup. Consider a vertex of the form



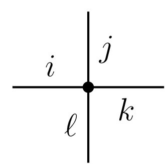
Each label i, j, k, l arrow pointing toward or away from the vertex. Denote arrows $\uparrow$ and $\leftarrow$ by $+$ and arrows $\downarrow$ and $\rightarrow$ by $-$. Then any quadruple of $+$ and $-$ determines a configuration of the vertex above. Let us read an quadruple from left to right and start decorating the edges of the vertex at ℓ and moving clockwise. For example, $- + -+$ corresponds to



the NE configuration. If we choose the ordered basis $(++, +-, -+, --)$, then the R-matrix

$$R = \begin{pmatrix} x_i & 0 & 0 & 0 \\ 0 & 0 & x_i & 0 \\ 0 & 1 & x_i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

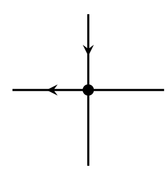
encodes all the weight data at a vertex. Observe we have put the entry 0 if the configuration doesn't correspond to a Boltzmann weight configuration. In particular, there are 5 nonzero entries corresponding to the nonzero Boltzmann weights. Let V be the two-dimensional vector space with basis vectors v_+ and v_-. Then we can think of



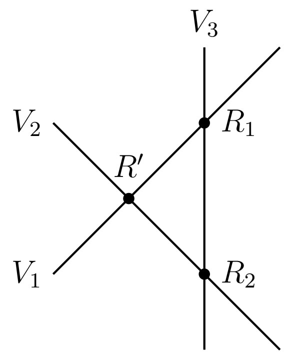
as an operator $V_i \otimes V_j \to V_k \otimes V_\ell$ where the operator is given by

$$v_i \otimes v_j \to R_{i,j}^{k,l}(v_k \otimes v_l)$$

and $R_{i,j}^{k,l}$ is the weight corresponding to the configuration i, j, k, l. Thus



is $x_i(v_+ \otimes v_-) + x_i(v_- \otimes v_+)$ because the third column of R has x_i as the $+-$ and $-+$ entry. With this viewpoint in mind, the diagram below is an element of $\mathrm{End}(V_1 \otimes V_2 \otimes V_3)$:



The segments of the edges closest to V_1, V_2, and V_3 are inputs and when we pass a vertex by moving left the inputs are acted upon by matrices R', R_1, and R_2 respectively (R' is the unknown weight matrix and R_1 and R_2 are R with $i = 1, 2$ respectively). The matrices only act on the vector spaces associated to the lines which intersect at their vertex (so R' acts on V_1 and V_2 and acts as the identity on V_3). Then the YBE is expressed as

$$R'R_1R_2 = R_2R_1R'$$

where the matrices are 8×8. Computing the products and solving the linear equations in the entries gives, the unknown 4×4 weight matrix R' is

$$R' = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{x_1}{x_2} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

for a parameter λ. □

1.3 The Problem

There are three parts to this problem:

1. Catalog identities satisfied by Schur functions and determine which ones have lattice model proofs.

2. Evaluate a partition function of lattice models in earlier work of Brubaker-Schultz.

3. Explore lattice models for k-Schur functions.

2 Birational R-matrices

2.1 Theory

Let M be an $n \times n$ matrix. A minor of M is $\Delta(M)_{I,J} := \det(M_{I,J})$ where $M_{I,J}$ is the submatrix of all entries of M in a row indexed by I and a column indexed by J. In particular, I and J need be the same size.

Example 2.1. Let $n = 3$ and consider the matrix

$$M = \begin{pmatrix} 2 & 2 & 0 \\ 6 & 12 & 9 \\ 0 & 0 & 2 \end{pmatrix}.$$

Then

$$\Delta(M)_{\{1,3\},\{1,2\}} = \det \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} = 0.$$

We say that M is totally positive if every minor is positive and we say M is totally nonnegative if every minor is nonnegative.

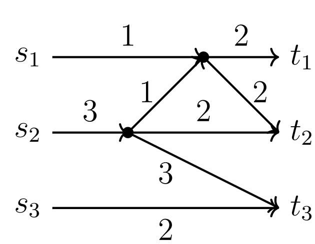
Example 2.2. Consider our matrix

$$M = \begin{pmatrix} 2 & 2 & 0 \\ 6 & 12 & 9 \\ 0 & 0 & 2 \end{pmatrix}.$$

Observe M is not totally positive since $\Delta(M)_{\{1,3\},\{1,2\}} = \det \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} = 0$. However, M turns out to be totally nonnegative (this required several more checks to verify).

We would like to now consider planar, directed, acyclic, edge-weighted graphs with n sources and n sinks such that the sources are sinks are separated. We will call these planar networks.

Example 2.3. Let $n = 3$. Then

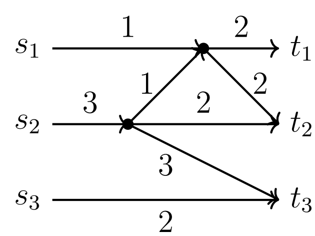


is an example of a planar network.

For any directed path from a source to a sink, the weight of the path is the product of the weights of the edges. The weight matrix is the matrix $M = (m_{i,j})$ where

$$m_{i,j} = \sum_{\text{paths } P: s_i \to t_j} \text{wt}(P).$$

Example 2.4. The weight matrix of



is

$$M = \begin{pmatrix} 2 & 2 & 0 \\ 6 & 12 & 9 \\ 0 & 0 & 2 \end{pmatrix}.$$

We have the following lemma:

Lemma 2.1. The weight matrix of a planar network is a totally nonnegative matrix. In particular

$$\det(M_{I,J}) = \sum_{\mathcal{F}} \prod_{P \in \mathcal{F}} \text{wt}(P)$$

where $\mathcal{F}$ is a family of nonintersecting paths from the sources indexed by I to sinks indexed by J and $P \in \mathcal{F}$ is a path in $\mathcal{F}$.

The converse is also true:

Theorem 2.1. Every nonnegative matrix is the weight matrix of a planar network.

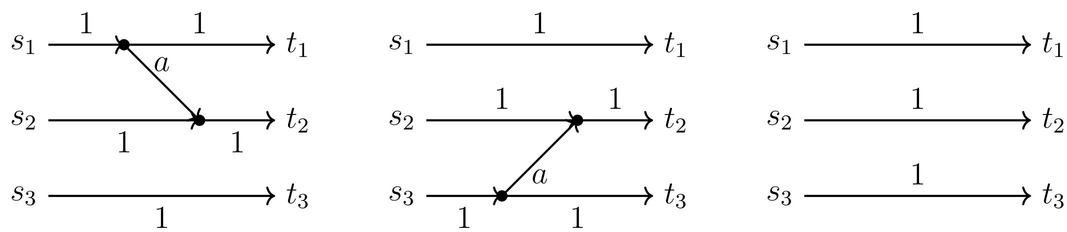
We also have an interesting theorem of Loewner-Whitney:

Theorem 2.2 (Loewner-Whitney)**.** Any invertible totally nonnegative matrix is a product of elementary Jacobi matrices with nonnegative matrix entries.

The elementary Jacobi matrices are described as follows:

- $e_i(a)$ is the identity with a in the $(i, i+1)$-entry.
- $f_i(a)$ is the identity with a in the $(i+1, i)$-entry.
- $h_i(a)$ is the identity with a in the (i, i)-entry.

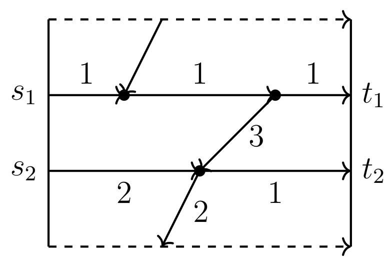
where a is any nonzero real number. The elementary Jacobi matrices $e_i(a)$, $f_i(a)$, and $h_i(a)$ correspond to the following simple planar networks



and concatenation of networks corresponds to multiplication of matrices. In particular, this shows that every element of $\mathrm{GL}_n(\mathbb{R})_{\geq 0}$ is the weight matrix of a planar network.

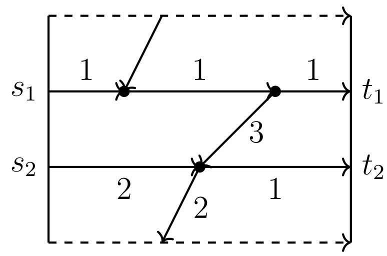
We would like to loosen the planar condition of our planar network by embedding them in a cylinder.

Example 2.5. Such a network we would like to consider takes the form



We call the dashed boundary a chord $\mathfrak{h}$. The network is acyclic in the sense that there are no cycles when the network is drawn on the universal cover of the cylinder. There is an associated weight matrix for this network built as follows: the (i, j) entry is the weight is the sum of all paths from source i to sink j but every time we pass over the chord $\mathfrak{h}$ from the bottom we multiply that weight by t and every time we pass over the chord $\mathfrak{h}$ from the top we multiply by t^{-1}. Then the entries of the weight matrix become Laurent series in t.

Example 2.6. The weight matrix M for the network



is given by

$$M = \begin{pmatrix} 1 + 6t + 36t^2 + \cdots & 3 + 18t + 108t^2 + \cdots \\ 4t + 24t^2 + \cdots & 2 + 12t + 72t^2 + \cdots \end{pmatrix}.$$

To each weight matrix for our cylindrical network we can associated an unfolding of the matrix which is an infinite array described as follows. The array contains repeated block matrices with with entries the constant terms in entries of M, t terms in the entries of M, t^{-1} terms in the entries of M, etc. As you move from left to right and down to up you increase by a power of t.

Example 2.7. The unfolding of the weight matrix

$$M = \begin{pmatrix} 1 + 6t + 36t^2 + \cdots & 3 + 18t + 108t^2 + \cdots \\ 4t + 24t^2 + \cdots & 2 + 12t + 72t^2 + \cdots \end{pmatrix}$$

is given by the array

$$\begin{array}{ccccccccc} \ddots & & & & & & & & \cdot^{\cdot^{\cdot}} \\ & 1 & 3 & 6 & 18 & 36 & 108 & & \\ & 0 & 2 & 4 & 12 & 24 & 72 & & \\ & 0 & 0 & 1 & 3 & 6 & 18 & & \\ & 0 & 0 & 0 & 2 & 4 & 12 & & \\ \cdot^{\cdot^{\cdot}} & & & & & & & & \ddots \end{array}$$

We have an analogous lemma which says the following:

Lemma 2.2. The unfolding of a weight matrix of a cylindrical network is totally nonnegative. In particular

$$\det(M_{I,J}) = \sum_{\mathcal{F}} \prod_{P \in \mathcal{F}} \mathrm{wt}(P)$$

where $\mathcal{F}$ is a family of nonintersecting paths from the sources indexed by I to sinks indexed by J in the universal cover of the network and $P \in \mathcal{F}$ is a path in $\mathcal{F}$.

In general however, the converse is not true. A totally nonnegative infinite periodic matrix might require an infinite network to construct it.

Let $\mathrm{GL}_n(\mathbb{R}(t))$ be the set of $n \times n$ matrices with entries that are formal Laurent series in t and that have nonzero determinant. Let $U \in \mathrm{GL}_n(\mathbb{R}(t))$ be the elements with unitriangular unfoldings. We define $U_{\geq 0} \subset U$ to be the elements of U with totally nonnegative unfoldings and $U_{>0} \subset U$ to be the elements of U with totally positive unfoldings in the sense that all minors not forced to be zero (some are forced by the Laurent series) are positive. Then we have the following theorem

Theorem 2.3. Every element of $U_{\geq 0} - U_{>0}$ is the weight matrix of a cylindrical network.

We will end our discussion with whirls and curls. The whirl $M(a_1, a_2, a_3)$ is defined by

$$M(a_1, a_2, a_3) := \begin{pmatrix} 1 & a_1 & 0 \\ 0 & 1 & a_2 \\ a_3 t & 0 & 1 \end{pmatrix}.$$

The curl $N(a_1, a_2, a_3)$ is defined by

$$N(a_1, a_2, a_3) := \left(\sum_{k=0}^{\infty} (a_1 a_2 a_3)^k\right) \begin{pmatrix} 1 & a_1 & a_1 a_2 \\ a_2 a_3 t & 1 & a_2 \\ a_3 t & a_1 a_3 t & 1 \end{pmatrix}.$$

In general, we can construct whirls and curls for n parameters not just 3. With these definitions we have the theorem:

Theorem 2.4. Any element of $U_{\geq 0} - U_{>0}$ is a product of whirls and curls with nonnegative parameters.

Such a decomposition is not unique in general but there is a relationship between different decompositions in certain cases.

Let $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ be elements of $\mathbb{R}^n_{\geq 0}$. Let

$$\kappa_i(\mathbf{a}, \mathbf{b}) = \sum_{j=i}^{i+n-1} \prod_{k=i+1}^{j} b_k \prod_{k=j+1}^{i+n-1} a_k,$$

and define η as the map that sends $(\mathbf{a}, \mathbf{b})$ to $(\mathbf{b}', \mathbf{a}')$ where

$$b'_i = \frac{b_{i+1} \kappa_{i+1}(\mathbf{a}, \mathbf{b})}{\kappa_i(\mathbf{a}, \mathbf{b})} \quad \text{and} \quad a'_i = \frac{a_{i-1} \kappa_{i-1}(\mathbf{a}, \mathbf{b})}{\kappa_i(\mathbf{a}, \mathbf{b})}.$$

and the indices are taken cyclically. This is called the birational R-matrix formula. We have the following theorem:

Theorem 2.5. The following properties hold:

1. $M(\mathbf{a})M(\mathbf{b}) = M(\mathbf{b}')M(\mathbf{a}')$ and $N(\mathbf{b})N(\mathbf{a}) = N(\mathbf{a}')N(\mathbf{b}')$.

2. η is an involution.

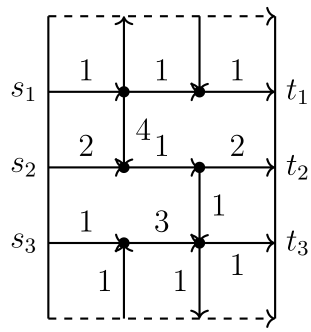
3. η satisfies the braid relation

$$\eta_i \circ \eta_{i+1} \circ \eta_i(\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(k)}) = \eta_{i+1} \circ \eta_i \circ \eta_{i+1}(\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(k)})$$

where η_i means apply η to the i and $i+1$ entry.

2.2 Exercises

1. Consider the cylindrical network N:



   (a) Compute the weight matrix of N.

   (b) Compute the unfolding of the weight matrix of N.

   (c) Check that unfolding of the weight matrix of N is totally nonnegative.

Proof.

(a) This will be a 3×3 matrix M. It is easily computed as:

$$M = \begin{pmatrix} 4t+4 & 8 & 3t^{-1}+4 \\ 2t & 4 & 2 \\ 3t & 0 & 3 \end{pmatrix}.$$

(b) The unfolding of the weight matrix N is then

$$\begin{array}{cccccccc} \ddots & & & & & & & \iddots \\ & 4 & 8 & 4 & 4 & 0 & 0 & \\ & 0 & 4 & 2 & 2 & 0 & 0 & \\ & 0 & 0 & 3 & 3 & 0 & 0 & \\ & 0 & 0 & 3 & 4 & 8 & 4 & \\ & 0 & 0 & 0 & 0 & 4 & 2 & \\ & 0 & 0 & 0 & 0 & 0 & 3 & \\ \iddots & & & & & & & \ddots \end{array}$$

(c) This is just checking that all possible minors are nonnegative which can best be done with compute software. Observe that we only need to check a few minors because we can write the array an as array of 3×3 block matrices and if we view the array in this form, everything off the super and sub diagonal is the zero matrix.

□

2.

(a) Compute $\eta(\mathbf{a}, \mathbf{b})$ when $\mathbf{a} = (1, 2, 3)$ and $\mathbf{b} = (2, 3, 4)$.

(b) Verify that $M(\mathbf{a})M(\mathbf{b}) = M(\mathbf{b}')M(\mathbf{a}')$ and $N(\mathbf{b})N(\mathbf{a}) = N(\mathbf{a}')N(\mathbf{b}')$ hold for this choice of $\mathbf{a}$ and $\mathbf{b}$.

Proof.

(a) We easily compute $\kappa_1(\mathbf{a}, \mathbf{b}) = 27$, $\kappa_2(\mathbf{a}, \mathbf{b}) = 15$, and $\kappa_3(\mathbf{a}, \mathbf{b}) = 12$. From this we easily compute $(\mathbf{a}', \mathbf{b}')$:

$$\mathbf{a}' = \left(\frac{4}{3}, \frac{9}{5}, \frac{5}{2}\right) \quad \text{and} \quad \mathbf{b}' = \left(\frac{5}{3}, \frac{16}{5}, \frac{9}{2}\right).$$

(b) All we need to do is some matrix multiplication. One the one hand,

$$M(\mathbf{a})M(\mathbf{b}) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3t & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 4t & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 3 \\ 8t & 1 & 5 \\ 7t & 6t & 1 \end{pmatrix}.$$

On the other hand

$$M(\mathbf{b}')M(\mathbf{a}') = \begin{pmatrix} 1 & \frac{5}{3} & 0 \\ 0 & 1 & \frac{16}{5} \\ \frac{9}{2}t & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{4}{3} & 0 \\ 0 & 1 & \frac{9}{5} \\ \frac{5}{2}t & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 3 \\ 8t & 1 & 5 \\ 7t & 6t & 1 \end{pmatrix}.$$

Both of these matrices are the same. The computation is similar for curls since we can ignore the coefficients

$$\left(\sum_{k=0}^{\infty}(a_1a_2a_3)^k\right) \quad \text{and} \quad \left(\sum_{k=0}^{\infty}(a_1'a_2'a_3')^k\right)$$

because $a_1a_2a_3 = a_1'a_2'a_3'$ and similarly for $\mathbf{b}$ and $\mathbf{b}'$.

□

3. Verify that Theorem 2.5 holds when $n = 2$.

Proof. We will verify each part of Theorem 2.5 separately.

(a) Let $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$. Then $\kappa_1(\mathbf{a}, \mathbf{b}) = a_2 + b_2$ and $\kappa_2(\mathbf{a}, \mathbf{b}) = a_1 + b_1$, so that

$$\mathbf{a}' = \left(\frac{a_2(a_1 + b_1)}{(a_2 + b_2)}, \frac{a_1(a_2 + b_2)}{(a_1 + b_1)} \right) \quad \text{and} \quad \mathbf{b}' = \left(\frac{b_2(a_1 + b_1)}{(a_2 + b_2)}, \frac{b_1(a_2 + b_2)}{(a_1 + b_1)} \right).$$

Now on the one hand

$$M(\mathbf{a})M(\mathbf{b}) = \begin{pmatrix} 1 & a_1 \\ a_2 t & 1 \end{pmatrix} \begin{pmatrix} 1 & b_1 \\ b_2 t & 1 \end{pmatrix} = \begin{pmatrix} 1 + a_1 b_2 t & a_1 + b_1 \\ (a_2 + b_2)t & 1 + a_2 b_1 t \end{pmatrix}.$$

On the other hand

$$\begin{aligned} M(\mathbf{b}')M(\mathbf{a}') &= \begin{pmatrix} 1 & \frac{b_2(a_1+a_2)}{(a_2+b_2)} \\ \frac{b_1(a_2+b_2)}{(a_1+b_1)} t & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{a_2(a_1+b_1)}{(a_2+b_2)} \\ \frac{a_1(a_2+b_2)}{(a_1+b_1)} t & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 + a_1 b_2 t & a_1 + b_1 \\ (a_2 + b_2)t & 1 + a_2 b_1 t \end{pmatrix}. \end{aligned}$$

The matrices agree. A similar calculation occurs for the curl matrices.

(b) We will show a_1 is fixed by η^2 as the computation is similar in the other cases. Observe $\kappa_1(\mathbf{a}', \mathbf{b}') = a_2 + b_2$ and $\kappa_2(\mathbf{a}', \mathbf{b}') = a_1 + b_1$. So

$$\eta^2(a_1) = \eta \left(\frac{a_2(a_1 + b_2)}{(a_2 + b_2)} \right) = \frac{a_1(a_2 + b_2)}{(a_1 + b_1)} \frac{(a_1 + b_2)}{(a_2 + b_2)} = a_1.$$

(c) All that is involved is a series of tedious, but straight forward calculations.

□

4. For $n = 2$ and $n = 3$, compute formulas for the actions of (123), (132), and (13)

Proof. The easiest way to compute these formulas is by compute software. □

2.3 The Problem

The birational R-matrix formula is a formula for how transpositions act on factorizations. Find a combinatorial formula for how the other elements of the symmetric group act.

3 Cluster Algebras

3.1 Theory

Let k be a field of characteristic zero. A cluster algebra $\mathcal{A}$ (of geometric type) is a subalgebra of $k[x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}]$ described as follows: specify an initial set of variables $\{x_1, \ldots, x_{n+m}\}$ called a cluster. The elements are called cluster variables. We then construct other cluster variables by binomial exchange relations

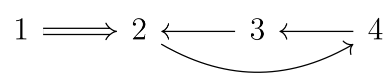
$$x_\alpha x'_\alpha := \prod x_{\gamma_i}^{d_i^+} + \prod x_{\gamma_i}^{d_i^-}$$

to be described in the following. Notice the x'_α isn't a new variable, its a Laurent polynomial in the $x_1, \ldots, x_{n+m}$. The algebra generated by these This binomial exchange relations corresponding to the cluster $\{x_1, \ldots, x_{n+m}\}$ together with the cluster $\{x_1, \ldots, x_{n+m}\}$ is called the seed.

We say Q is a quiver if it is a directed graph with loops and multiple edges allowed. If Q is a quiver with vertices corresponding to $x_1, \ldots, x_{n+m}$, then the binomial exchange relations are

$$x_j x'_j := \prod_{i \to j \in Q} x_i + \prod_{j \to k \in Q} x_k.$$

Example 3.1. Let Q be the quiver



Then

$$x_1 x'_1 = 1 + x_2^2 \qquad x_2 x'_2 = x_1^2 x_3 + x_4$$
$$x_3 x'_3 = x_4 + x_2 \qquad x_4 x'_4 = x_2 + x_3$$

In general, if Q has n vertices we obtain n new seeds starting from the initial seed by mutating the quiver in n directions. To describe this mutation process, let Q be a quiver and j be a vertex of the quiver. Then $Q' = \mu_j Q$ the mutation of Q at j is formed by applying the following steps:

1. For any path

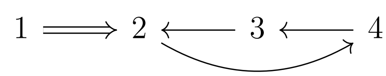
$$i \longrightarrow j \longrightarrow k$$

in Q we add a new arrow

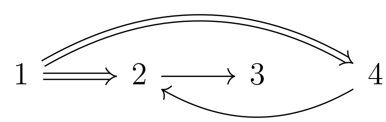
2. Reverse the direction of all arrows incident to j.

3. Delete any two cycles created from the previous two steps.

Example 3.2. If Q is the quiver



then $\mu_2 Q$ is



Mutation is an involution; $\mu_j^2 Q = Q$ for any vertex j. We can build an $n \times n$ skew-symmetric matrix $B_Q[b_{ij}]$ whose entries b_{ij} are the difference between the number of arrows $i \to j$ and $j \to i$.

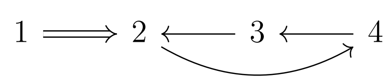
Example 3.3. If Q is the quiver

$$1 \longrightarrow 2$$

then

$$B_Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

If Q is the quiver



then

$$B_Q = \begin{pmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix}.$$

We can also mutate the matrix B_Q at a vertex k, where $\mu_k B_Q = [b'_{ij}]$ is given by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + [b_{ik}]_+[b_{jk}]_+ - [-b_{ik}]_+[-b_{kj}]_+ & \text{otherwise} \end{cases}$$

where $[\alpha]_+ = \max(\alpha, 0)$.

Example 3.4. If we have

$$B_Q = \begin{pmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix},$$

then

$$\mu_4 B_Q = \begin{pmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}.$$

Given a quiver Q on n vertices and its associated matrix B_Q, we can build a corresponding $2n \times n$ exchange matrix $\tilde{B}_Q$ defined by

$$\tilde{B}_Q = \begin{pmatrix} B_Q \\ I_n \end{pmatrix}$$

where I_n is the $n \times n$ identity matrix. In this sense, $\tilde{B}_Q$ is the $n \times n$ matrix corresponding to the quiver $\tilde{Q} = Q \cup \{1', 2', \ldots, n'\}$ with a single arrow $i' \to i$ for $1 \leq i \leq n$ and all original arrows in the quiver Q.

Now suppose we start with a framed quiver $\tilde{Q} = Q \cup \{1', 2', \ldots, n'\}$ and an initial cluster $\{x_1, \ldots, x_N\} = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$. We can iterate mutations with extra restrictions by disallowing mutations at vertices i'. The binomial exchange relations are then

$$x'_k = \frac{\prod_{i=1}^n x_i^{[b_{ik}]_+} + \prod_{k=1}^n x_i^{[-b_{ik}]_+}}{x_k} = \frac{\prod_{i \to k} x_i + \prod_{k \to i} x_i}{x_k}.$$

It is the case that the y_i never appear in the denominator. By letting $x_1 = x_2 = \cdots = x_n = 1$, and iterating the mutation, we replace cluster variables (which are Laurent polynomials in the x_i and y_i) with polynomials in the y_i. These are called F-polynomials. We define c-vectors associated to the seed t of $\tilde{Q}$ by

$$\mathbf{c}_{j,t} = [c_{1j}, c_{2j}, \ldots, c_{nj}]^{\mathrm{T}}$$

where c_{ij} is the number of arrows $i' \to j$. Sometime we suppress the subscript t if the seed is understood. Equivalently $\mathbf{c}_{j,t}$ is the jth column of the bottom half of the $2n \times n$ exchange matrix. The initial vectors for the starting seed t_0 are the unit vectors, and they mutate as we mutate the seed. In particular, if

$$\mathbf{c}_{j,\mu_k t} = [c'_{1j}, c'_{2j}, \ldots, c'_{nj}]^{\mathrm{T}}$$

then

$$c'_{ij} = \begin{cases} -c_{ij} = -c_{ik} & \text{if } j = k, \\ c_{ij} + [c_{ik}]_+[b_{kj}]_+ - [-c_{ik}]_+[-b_{kj}]_+ & \text{otherwise.} \end{cases}$$

There is a particularly simple but difficult to prove theorem regarding the entries of the c-vectors:

Theorem 3.1 (Derksen-Weyman-Zelevinsky)**.** Each c-vector consists of all nonnegative or all nonpositive entries.

We also have another important theorem:

Theorem 3.2. Given a framed quiver $\tilde{Q}$ and a mutation sequence $\bar{u} = \mu_{i_1}\mu_{i_2}\cdots\mu_{i_\ell}$, consider the sequence of cluster seeds

$$t_0 \xrightarrow{\mu_{i_1}} t_1 \xrightarrow{\mu_{i_2}} t_2 \xrightarrow{\mu_{i_3}} \cdots \xrightarrow{\mu_{i_\ell}} t_\ell.$$

Let $L_1 = 1 + z_1$ and

$$L_k = 1 + z_{k_\ell} L_1^{\mathbf{c}_1 B_Q |\mathbf{c}_k|} L_2^{\mathbf{c}_2 B_Q |\mathbf{c}_k|} \cdots L_1^{\mathbf{c}_{k-1} B_Q |\mathbf{c}_k|}$$

for $k \geq 2$. Then

$$F_{i_\ell;t_\ell} = \prod_{j=1}^{\ell} L_j^{\mathbf{c}_j \mathbf{g}_\ell}|_{z_1 = y^{|\mathbf{c}_1|}, z_2 = y^{|\mathbf{c}_2|}, \ldots, z_\ell = y^{|\mathbf{c}_\ell|}}$$

where $\mathbf{g}_\ell$ is the exponent vector of the monomial in the cluster variables that only contains the y_i and $\mathbf{c}_p$ (respectively $|\mathbf{c}_p|$ or $\mathbf{g}_p$) denotes the pth c-vector along the mutation sequence $\bar{\mu}$.

We will now switch gears and discuss q-binomial coefficients. We define q-binomial coefficients by the formula

$$\begin{bmatrix} n+k \\ k \end{bmatrix}_q := \frac{(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{n+k})}{(1-q)(1-q^2)\cdots(1-q^k)}.$$

There is another formula for computing q-binomial coefficients called the Kathleen O'Hara formula (KOH):

$$\begin{bmatrix} n+k \\ k \end{bmatrix}_q = \sum_{\lambda} q^{2n(\lambda)} \prod_{i=0}^{k-1} \begin{bmatrix} (k-i)n - 2i + (**) \\ m_{k-i} \end{bmatrix}_q$$

where the sum is over all partition λ of k, m_i is the number of occurrences of i in λ,

$$n(\lambda) = \sum_{i=1}^{k} (i-1)\lambda_i,$$

and

$$(**) = \sum_{j=0}^{i-1} 2(i-j)m_{k-j} + m_{k-i}.$$

Recall that the Fibonacci numbers F_n can be decomposed into sums of binomial coefficients:

$$F_n = \sum_{k=1} \binom{n-k}{k-1}.$$

We define the Carlitz q-Fibonacci numbers $F_n(q)$ by

$$F_n(q) := \sum_{k=1}^{n-k+1} q^{(k-1)^2} \begin{bmatrix} n-k \\ k-1 \end{bmatrix}_q,$$

and a variant set of numbers $\tilde{F}_n(q)$ by

$$\tilde{F}_n(q) := \sum_{k=1}^{n-k+1} q^{(k-1)} \begin{bmatrix} n-k \\ k-1 \end{bmatrix}_q.$$

3.2 Exercises

1. Let $\mathcal{A}$ be the cluster algebra defined by the initial cluster $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ with initial binomial exchange relations

$$x_1 x_1' = y_1 + y_2, \quad x_2 x_2' = x_1 x_3 y_2 + 1, \quad \text{and} \quad x_3 x_3' = y_3 + x_2.$$

So

$$B_Q = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Show that each seed of $\mathcal{A}$ corresponds to a triangulation of a hexagon such that chords correspond to cluster variables. Furthermore, show $\mathcal{A}$ is of finite type and determine all the cluster variables.

2. Use the generalized binomial theorem and the product expansion for $F_{i_\ell;t_\ell}$ to derive the power series expansion

$$F_{i_\ell,t_\ell} = \sum_{(m_1,\ldots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{j=1}^{\ell} \binom{\mathbf{c}_j \cdot (*)}{m_j} \mathbf{y}^{\sum_{j=1}^{\ell} m_j |\mathbf{c}_j|}$$

where

$$(*) = \mathbf{g}_\ell + \sum_{k=j+1}^{\ell} m_k B_Q |\mathbf{c}_k|.$$

3.
   (a) Compute the q-Fibonacci numbers $F_n(q)$ for $3 \leq n \leq 7$.
   (b) Prove that for $n \geq 3$, $F_n(q) = F_{n-1}(q) + q^{n-2} F_{n-2}(q)$.
   (c) Give and prove a combinatorial interpretation for $F_n(q)$ in terms of counting integer partitions.

4.
   (a) Compute $\tilde{F}_n(q)$ for $3 \leq n \leq 7$.
   (b) Describe a combinatorial interpretation for the $\tilde{F}_n(q)$.
   (c) Describe a $\mathbb{Z}[q, q^{-1}]$-specialization of the F-polynomials for the Kronecker quiver such that for each $\ell \geq 3$, we have $F_{i_\ell;t_\ell}$ specializing to $\tilde{F}_{2\ell+1}(q)$.

3.3 The Problem

There are a suite of problems:

1. Develop a (q, t)-analogue of the KOH formula for binomial coefficients and identify the associated algebraic transformation such that the analogous sum of (q, t)-binomial coefficients match the formulas for $F_{i_\ell;t_\ell}(y_1, y_2)$ for the Kronecker quiver.

2. Explicitly demonstrate positivity and polynomiality of these power series expressions. Describe how to regroup terms of this power series to match up with known combinatorial formulas for cluster variables of F-polynomials in the rank two case.

3. Develop power series formulas for missing g-vectors beyond the case of the Kronecker quiver.

4 Sequences from Cylindrical Graphs

4.1 Theory

We say a sequence $a_0, \ldots, a_n$ of real numbers is unimodal if for some $s \in [n]$, we have

$$a_0 \leq \cdots \leq a_s \geq a_{s+1} \geq \cdots \geq a_n.$$

That is, there is a single peak in the sequence at a_s.

Example 4.1. Many common combinatorial sequences are unimodal:

- $\binom{n}{k}$ is unimodal in k for fixed n.
- $S(n,k)$ is unimodal in k for fixed n.
- $c(n,k)$ is unimodal in k for fixed n.
- Let $K_{\lambda,\mu}$ be the number of SSYT of shape λ and weight μ. Set

  $$a_{ij} = (0, \ldots, 1, 0 \ldots, -1, 0, \ldots, 0)$$

  with 1 in the ith position and -1 in the jth position. Then $K_{\lambda,\mu+k\alpha_{ij}}$ in unimodal in k.

We call a sequence log-concave if $a_i^2 \geq a_{i-1}a_{i+1}$ for all i. The reason we give such a sequence this name is because if we take the logarithm of the sequence, it is concave in the calculus sense.

A method to prove unimodality of a sequence is to construct sets T_i such that $|T_i| = a_i$ and show that there are injections $T_i \hookrightarrow T_{i+1}$ for $0 \leq i \leq s-1$ and injections $T_i \hookrightarrow T_{i-1}$ for $s+1 \leq i \leq n$. Similarly, a method to prove log-concavity of a sequence is to construct analogous injections $T_{i-1} \times T_{i+1} \hookrightarrow T_i \times T_i$ for $0 \leq i \leq n$.

We say that a sequence $(a_i)_{i=0}^n$ in $\mathbb{R}_{\geq 0}$ is a Pólya Frequency sequence (PFS) if its generating function

$$\sum_{i=1}^{n} a_i t^i$$

has only real roots. We have a nice theorem:

Theorem 4.1 (Edrei-Thoma)**.** $(a_i)_{i=0}^n$ is a PFS if and only if the infinite array

$$\begin{array}{ccccccccc}
\ddots & & & & & & & & \iddots \\
& a_0 & a_1 & \cdots & a_n & 0 & 0 & 0 & \\
& 0 & a_0 & a_1 & \cdots & a_n & 0 & 0 & \\
& 0 & 0 & a_0 & a_1 & \cdots & a_n & 0 & \\
& 0 & 0 & 0 & a_0 & a_1 & \cdots & a_n & \\
\iddots & & & & & & & & \ddots
\end{array}$$

is totally nonnegative.

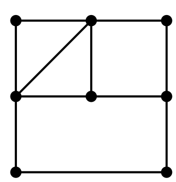
For any PFS, the inequality

$$\left(\frac{a_k}{\binom{n}{k}}\right)^2 \geq \frac{a_{k-1}}{\binom{n}{k-1}} \frac{a_{k+1}}{\binom{n}{k+1}},$$

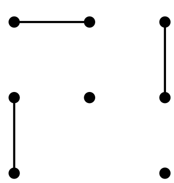
called Newton's inequality, holds for every term in the sequence. It's not obvious, but this is a stronger inequality that log-concave and unimodal. There is also a nice method to prove a sequence is PFS. The idea is to give a combinatorial interpretation of the minors of the infinite array described in the Edrei-Thoma theorem as natural numbers. Since the determinant is a signed sum this amounts to showing that the negative terms cancel with corresponding positive terms and then understanding what the remaining positive terms count.

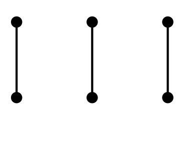
Let G be a undirected graph. We call an edge in G a dimer. A dimer configuration is a collection of edges in G using each vertex either 0 or 1 times. A dimer cover is a dimer configuration such that every vertex is used exactly once.

Example 4.2. For example, consider the undirected graph



Then the graphs below are a dimer configuration and a dimer cover respectively:



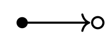


There is a nice theorem:

Theorem 4.2 (Heilmann-Lieb)**.** For any G, let a_k be the number of dimer configurations on G using k dimers. Then (a_k) is a PFS.

We would now like to work with these graphs on infinite cylinders. Let the counterclockwise orientation on the cylinder be the positive orientation and let G be a bipartite graph embedded in the cylinder. With vertices one of the two forms

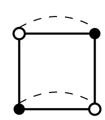
If π is a dimer cover we will by convention assume the edges are directed



We will denote by $\pi^\vee$ the same dimer cover but with the edges directed in the opposite fashion. Then for all dimer covers π_1 and π_2, $\pi_1 \cup \pi_2^\vee$ is a union of oriented simple cycles possibly with two cycles. Observe that some cycles may be contractible. We define the height of $\pi_1 \cup \pi_2^\vee$, denoted $\mathrm{ht}(\pi_1, \pi_2)$, to be the difference between the number of positively oriented cycles in in $\pi_1 \cup \pi_2^\vee$ and negatively oriented cycles in $\pi_1 \cup \pi_2^\vee$. Think of $\mathrm{ht}(\pi_1, \pi_2)$ as the homology class or total winding number of $\pi_1 \cup \pi_2^\vee$. It turns out that for all dimer covers π_1, π_2, and π_3, that

$$\mathrm{ht}(\pi_1, \pi_2) = \mathrm{ht}(\pi_1, \pi_3) - \mathrm{ht}(\pi_2, \pi_3).$$

Example 4.3. Consider the graph G embedded in the cylinder



It admits to dimer covers π_1 and π_2 respectively:

Then we have $\mathrm{ht}(\pi_1, \pi_2) = 1$.

4.2 Exercises

1. If $(a_i)_{i=0}^n$ is a log-concave sequence in $\mathbb{R}_{>0}$, prove it is unimodal. Prove that the same holds if $(a_i)_{i=0}^n$ is a log-concave sequence in $\mathbb{R}_{\geq 0}$ and has no internal zeros.

   Proof. Denote by a_s the maximum element of the sequence. Then

   $$a_{s-1} \leq a_s \geq a_{s+1}.$$

   By log-concavity

   $$a_{s+1}^2 \geq a_s a_{s+2},$$

   and these two inequalities together imply $a_{s+1} \geq a_{s+2}$. A similar argument shows $a_{s-2} \leq a_{s-1}$. Repeating this procedure in both directions proves the sequence is unimodal. We can repeat this procedure with a log-concave $(a_i)_{i=0}^n$ in $\mathbb{R}_{\geq 0}$ as long as it has no internal zeros so that we never divide by zero. □

2. Prove that a PFS is log-concave with no internal zeros.

   Proof. It follows from the inequality

   $$k(n-k) \leq (k+1)(n-k+1)$$

   that the binomial coefficients $\binom{n}{k}$ are log-concave in k for fixed n. Now suppose $(a_i)_{i=0}^n$ in $\mathbb{R}_{\geq 0}$ is a PFS. Then for every term in the sequence we have

   $$\left(\frac{a_k}{\binom{n}{k}}\right)^2 \geq \frac{a_{k-1}}{\binom{n}{k-1}} \frac{a_{k+1}}{\binom{n}{k+1}}.$$

   Since the binomial coefficients are log-concave in k,

   $$a_k^2 \geq a_{k-1}a_{k+1}$$

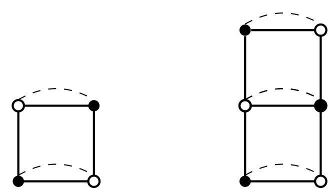
   which implies $(a_i)_{i=0}^n$ is log-concave. □

3. Explain why $\pi_1 \cup \pi_2^\vee$ is always a union of oriented simple cycles.

   Proof. Recall that a dimer cover π is a collection of edges where every vertex is used once. Consider the set of edges incident to vertex v across both dimer covers. Then the local picture at v is of one of the two forms:

If we are in the first case, then one edge points inward and the other outward in $\pi_1 \cup \pi_2^\vee$. In the second case, the 2-cycle is isolated in $\pi_1 \cup \pi_2^\vee$. Hence the local picture in the first case is involved in a simple cycle. All cycles are oriented because the cylinder is orientable. □

4. Identify all dimer covers and their relative heights for the graphs:



4.3 The Problem

In all cases the underlying graph G is a bipartite graph embedded in a cylinder. The problems are as follows:

1. Let a_k be the number of dimer covers of height k. Show that (a_k) is a PFS.

2. Give a combinatorial proof of the Heilmann-Lieb theorem.

3. Call a union of k vertex-disjoint simple cycles in G a k-cycle. Let a_k be the number of k-cycles in G. Give a combinatorial proof that (a_k) is a PFS.

4. Call a cycle rooted spanning forest (CRSF) a spanning subgraph whose connected components have the same number of vertices as edges. Call connected component essential if it has nonzero winding number. Let c_k be the number of CRSF in G with k essential connected components. Prove

$$\sum c_k \left(2 - t + \frac{1}{t}\right)^k$$

is real-rooted via a combinatorial proof of the Edrei-Thoma theorem.

5 Torics and Regularity

5.1 Theory

We will now assume all modules are left modules unless otherwise specified. Let $S = \mathbb{C}[x_1, \ldots, x_n]$ and let $f_1, \ldots, f_r$ be homogeneous polynomial in S. Consider the homogeneous ideal $I = \langle f_1, \ldots, f_r \rangle$. By construction $f_1, \ldots, f_r$ are a minimal set of generators for I, but this set need not be unique. It turns out that the maximal degree among elements of a minimal generating set is unique and we denote it by $\operatorname{maxdeg}(I)$. It is natural to ask if we can give nice upper bounds for $\operatorname{maxdeg}(I)$, but this turns out to be a very hard question. Another question which turns out to have a better answer is to ask if we can provide nice upper bounds for the regularity of I, denoted, $\operatorname{reg}(I)$, which will be introduced in the following. We first need to discuss resolutions of the quotient ring S/I. A resolution of S/I is an exact sequence

$$F : F_0 \xleftarrow{\varphi_1} F_1 \xleftarrow{\varphi_2} \cdots \xleftarrow{\varphi_n} F_n \leftarrow 0$$

of S-modules such that

$$H_i(F) := \frac{\ker \varphi_i}{\operatorname{im} \varphi_{i+1}} = \begin{cases} S/I & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

In this formula $\varphi_0 : F_0 \to 0$ is the zero map. We can construct resolutions as follows: let $F_0 = S$ and let φ_1 be the row vector with entries $f_1, \ldots, f_r$. This forces $H_0(F) = S/I$. Then construct F_1 and φ_2 such that the sequence is exact at F_1, and so on. This involves making choices for F_1 and φ_2 and so on, so resolutions of S/I are not unique. We say a resolution F of S/I is minimal if n is minimal among all resolutions of S/I. We can obtain a minimal resolution from a resolution by composing all maps φ_i with $i > n$ if necessary. Thus we may assume our resolutions F are minimal. Two resolutions F and F' are said to be isomorphic if we have a commutative diagram of the form

$$\begin{array}{ccc} F_i & \xleftarrow{\varphi_{i+1}} & F_{i+1} \\ \sim\big\downarrow & & \big\downarrow\sim \\ F'_i & \xleftarrow[\varphi'_{i+1}]{} & F'_{i+1} \end{array}$$

for $0 \leq i \leq n-1$. It turns out that minimal resolutions are unique up to isomorphism, so we may speak of the minimal resolution F of S/I.

Example 5.1. Let $I = \langle x^2, xy^2, y^3 \rangle$. Then

$$S \xleftarrow{\begin{pmatrix} x^2 & xy^2 & y^3 \end{pmatrix}} S(-2) \oplus S(-3)^2 \xleftarrow{\begin{pmatrix} y^2 & 0 \\ -x & y \\ 0 & -x \end{pmatrix}} S(-4)^2 \leftarrow 0$$

is a resolution over S/I. The module $S(-2)$ denotes a free S-module with one generator where the -2 keeps track of the fact that all elements in this module are raised by homogeneous degree 2 in $\operatorname{im} \varphi_1$; similarity for $S(-3)$, $S(-4)$, and so on.

To the minimal resolution F of S/I, there corresponds a a Betti table $\beta(S/I)$. We index the rows and columns of the Betti table with elements of $\mathbb{Z}_{\geq 0}$. The (i, j)-entry of $\beta(S/I)$ is the degree of $S(-(i+j))$ in F_i. This is best illustrated with an example:

Example 5.2. Let $I = \langle x^2, xy^2, y^3 \rangle$, and consider the resolution

$$S \xleftarrow{\begin{pmatrix} x^2 & xy^2 & y^3 \end{pmatrix}} S(-2) \oplus S(-3)^2 \xleftarrow{\begin{pmatrix} y^2 & 0 \\ -x & y \\ 0 & -x \end{pmatrix}} S(-4)^2 \leftarrow 0.$$

The Betti table is Observe that the table extends infinitely down and to the right,

$\beta(S/I)$	0	1	2
0	1	0	0
1	0	1	0
2	0	2	2

but all these entries are zero.

The regularity of I, denoted $\operatorname{reg}(I)$, is defined to be the number of nonzero rows of $\beta(S/I)$. The projective dimension of S/I, denoted $\operatorname{pdim}(S/I)$, is defined to be the number of nonzero maps in the minimal resolution F of S/I.

We will now switch gears, and discuss projective space. Define complex projective $n-1$ dimensional space $\mathbb{P}^{n-1}$ to be the quotient $\mathbb{C}^n - \{\mathbf{0}\}/\mathbb{C}^x$. That is, $\mathbb{P}^{n-1}$ is $\mathbb{C}^n - \{\mathbf{0}\}$ modded out by scalar multiples. Let $\pi : \mathbb{C}^n - \{\mathbf{0}\} \to P^{n-1}$ denote the projection map. Given a homogeneous polynomial $f \in S$, the value $f(\bar{\mathbf{x}})$ is in general not well-defined for $\bar{\mathbf{x}} \in \mathbb{P}^{n-1}$. However, it does make sense to ask if f is zero at $\bar{\mathbf{x}}$. With this in mind, given a homogeneous ideal $I \in S$, define the variety of I to be

$$V(I) := \{\bar{\mathbf{x}} \in \mathbb{P}^{n-1} \mid f(\bar{\mathbf{x}}) = 0 \text{ for all } f \in I\}.$$

We will call the set of points that vanish on all elements of the ideal I the zero-set of I. In particular, $V(I)$ is the zero-set of I in $\mathbb{P}^{n-1}$.

Example 5.3. Consider the ideal $I = \langle xz - y^2 \rangle$. The zero-set of this ideal in $\mathbb{R}^3$ is the unit cone centered about the y-axis. However, the zero-set of I in $\mathbb{P}^2$ is the unit circle.

Given a variety V, we define its dimension $\dim(V)$, to be the dimension of $\pi^{-1}(V)$ in the usual sense. Then we define $\dim(S/I)$ by

$$\dim(S/I) := 1 + \dim(V(I)).$$

The codimension of I, denoted $\operatorname{codim}(I)$, is defined by

$$\operatorname{codim}(I) := n - \dim(S/I).$$

It turns out that $\operatorname{pdim}(S/I) \geq \operatorname{codim}(I)$; we say I is Cohen-Macaulay (CM) if $\operatorname{pdim}(S/I) = \operatorname{codim}(I)$. In other words, I is CM if the projective dimension of S/I is as short as possible. We define the degree of S/I, denoted $\deg(S/I)$, to be the number of points in $V(I) \cap L$ where L is a hyperplane of dimension $\operatorname{codim}(I)$ not intersecting $V(I)$ trivially. It is well-known that $\deg(S/I)$ is independent of the hyperplane L chosen.

We now can state our first bound for the regularity of I:

Theorem 5.1. Let I be a homogeneous ideal. Then

$$\operatorname{reg}(I) \leq (2 \cdot \operatorname{maxdeg}(I))^{2^{n-2}}.$$

There was a conjecture by Eisenbud-Goto proposed in 1984 which stated that if I is prime and $I \subseteq \langle x_1, \ldots, x_n \rangle^2$, then

$$\operatorname{reg}(I) \leq \deg(S/I) - \operatorname{codim}(I) + 1.$$

It was proven that this bound holds when I is also CM. However, it was shown in 2018 that the Eisenbud-Goto fails in general.

Let A be a $d \times n$ matrix with entries in $\mathbb{Z}$ and let u be an $n \times 1$ vector with entries in $\mathbb{Z}$ such that $u \in \ker_{\mathbb{Z}} A$. Then we can decompose u into a sum of two n-tuples by parity: $u = u_+ - u_-$. Letting the colums of the matrix A correspond to the variables $x_1, \ldots, x_n$ we define the toric ideal of A, denoted I_A, by

$$I_A := \langle x^{u_+} - x^{u_-} \mid Au = 0 \rangle$$

where x^{u_+} and x^{u_-} are monomials in the variables $x_1, \ldots, x_n$.

Example 5.4. Suppose we are working in $\mathbb{C}[u, v, w, x, y, z]$, and consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

Then $\ker_{\mathbb{Z}} A$ is generated by the elements

$$\begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 2 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} -2 \\ 3 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then the toric ideal I_A is

$$I_A = \langle y^2 - zu, xu - w^2, v^3 - u^2 w \rangle.$$

If we label the columns of A by $a_1, \ldots, a_n$, we define the semigroup generated by A to be

$$\mathbb{N}A := \{m_1 a_1 + \cdots + m_n a_n \mid m_i \in \mathbb{Z}_{\geq 0}\}.$$

Moreover, let $\mathrm{vol}(A)$ be $d!$ times the volume of $\mathrm{conv}(A, \mathbf{0})$. This turns out to be the $\deg(I_A)$. The semigroup ring of A is

$$\mathbb{C}[\mathbb{N}A] := \bigoplus_{a \in \mathbb{N}A} \mathbb{C}t^a.$$

It is a subring of $\mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$. We have the following theorem:

Theorem 5.2. If I_A is a toric ideal, then

$$S/I_A \cong \mathbb{C}[\mathbb{N}A].$$

5.2 Exercises

1. Compute the minimal resolutions, Betti tables, projective dimensions, regularity, degree, and codimension of the following ideals:

   (a) $\langle x, y\rangle \subseteq \mathbb{C}[x, y]$.

   (b) $\langle x^2, xy, y^2, xw^2 + yz^2\rangle \subseteq \mathbb{C}[w, x, y, z]$.

   (c) $\langle wy, wz, xy, xz\rangle \subseteq \mathbb{C}[w, x, y, z]$.

   (d) $\langle w^2 + x^2, y^2 + z^2, wz - xy, wy - xz\rangle \subseteq \mathbb{C}[w, x, y, z]$.

   Then show that the hypotheses of the Eisenbud-Goto conjecture are necessary by checking that each of these ideal fails the conjecture but also fails to satisfy one of the hypotheses.

2. Prove that every quotient of a polynomial ring is isomorphic to the quotient of a polynomial ring by a trinomial ideal.

3.

   (a) Prove that a binomial $x^u - x^v$ is homogeneous if and only if the dot product of $\mathbf{1}$ and $u - v$ is zero.

   (b) If $g \in \mathrm{GL}_d(\mathbb{Z})$, prove $I_A \cong I_{gA}$.

   (c) Let $\deg(x_i) = a_i \in \mathbb{Z}^d$. Show that primes in S/I_A are in bijective correspondence with the faces of $\mathbb{R}_{\geq 0}A$.

4. Assume all ideals are toric ideals I_A with $A \in \mathbb{Z}^{d\times n}$.

   (a) Show that the Eisenbud-Goto conjecture holds for $n - d = 1$.

   (b) Show that the Eisenbud-Goto conjecture holds for $d = 2$, $n - d = 2$, and one missing condition.

   (c) Find Veronese rings for which the Eisenbud-Goto conjecture holds.

5.3 The Problem

The problem is to understand for which $A \in \mathbb{Z}^{d \times n}$ does equality hold in the Eisenbud-Goto conjecture.

6 Frieze Patterns

6.1 Theory

A finite Frieze patter is a bi-infinite (infinite to the left and right) array of integers such that the following properties are satisfied:

- Every other row is offset to the right by one.
- The first and last rows consists of only 0s.
- The second and second to last rows consists of only 1s.
- All diamonds of the form

$$\begin{matrix} & a & \\ b & & c \\ & d & \end{matrix}$$

  satisfy the diamond condition $bc - ad = 1$.

We will fix the following notation:

$$\begin{matrix}
0 & & 0 & & 0 & & 0 & & 0 & \\
& 1 & & 1 & & 1 & & 1 & & 1 \\
m_{-1,1} & & m_{0,2} & & m_{1,3} & & m_{2,4} & & m_{3,5} & \\
& m_{-1,2} & & m_{0,3} & & m_{1,4} & & m_{2,5} & & m_{3,7} \\
m_{-2,2} & & m_{-1,3} & & m_{0,4} & & m_{1,5} & & m_{2,6} & \\
& m_{-2,3} & & m_{-1,4} & & m_{0,5} & & m_{1,6} & & m_{2,7}
\end{matrix}$$

The width of the Frieze pattern is the number of rows. We call the first nontrivial row (a row not of only 0s or 1s) the quiddity row. The quiddity row determines the entire Frieze pattern by the diamond condition. Frieze patterns are glide symmetric meaning they are symmetric with respect to a translation and reflection. Any smallest repeating segment of the Frieze pattern is called a fundamental domain. It also determines the Frieze pattern. We call a Frieze pattern periodic if its quiddity row is periodic in the entries. We call a Frieze pattern infinite if it has infinitely many rows. In particular, this means the quiddity row is infinite.

Example 6.1. The pattern

$$\begin{array}{cccccccccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & 1 & & 3 & & 1 & & 3 & & 2 & & 2 & & 1 & & 5 & & 1 & & \\
 & & 2 & & 2 & & 2 & & 5 & & 3 & & 1 & & 4 & & 4 & & 2 & \\
\cdots & 7 & & 1 & & 3 & & 3 & & 7 & & 1 & & 3 & & 3 & & 7 & & \cdots \\
 & & 3 & & 1 & & 4 & & 4 & & 2 & & 2 & & 2 & & 5 & & 3 & \\
 & 1 & & 2 & & 1 & & 5 & & 1 & & 3 & & 1 & & 3 & & 1 & & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

is a finite periodic Frieze pattern of period 6. Its quiddity row is

$$1 \quad 3 \quad 1 \quad 3 \quad 2 \quad 2 \quad 1 \quad 5 \quad 1$$

and its fundamental domain is

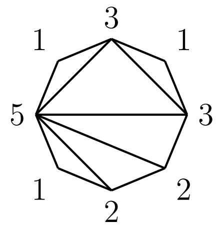
$$\begin{array}{ccccccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & & 1 & & 3 & & 2 & & 2 & & 1 & & 5 & & \\
 & & & 2 & & 5 & & 3 & & 1 & & 4 & & & \\
 & & & & 3 & & 7 & & 1 & & 3 & & & & \\
 & & & & & 4 & & 2 & & 2 & & & & & \\
 & & & & & & 1 & & 3 & & & & & & \\
 & & & & & & & 1 & & & & & & &
\end{array}$$

We have a result which connects Frieze patters and triangulations of polygons:

Theorem 6.1 (Conway-Coxeter)**.** *Finite Frieze patterns of positive integers with width n are in bijection with triangulations of $(n-1)$-gons.*

In the Conway-Coxeter theorem, the vertices of the $n-1$-gon correspond to the entries of the fundamental domain of the quiddity row such that the entry is the number of triangles incident to that vertex. This choice appears dependent on the order the vertices (clockwise or counterclockwise) but this of little matter since the Frieze pattern is glide symmetric.

Example 6.2. The triangulation



of the octagon corresponds to the fundamental domain

0 0 0 0 0 0 0 0
1 1 1 1 1 1 1
1 3 2 2 1 5
2 5 3 1 4
3 7 1 3
4 2 2
1 3
1

and hence the Frieze pattern it determines.

Frieze patterns can be described geometrically in a different way, but we need to first introduce some definitions. Let S_n be the punctured disc with n marked points on the boundary. Let $A_{n,m}$ be the annulus with n marked points on the outer boundary and m marked points on the inner boundary. We call an arc peripheral if it goes between points on the same boundary. We call an arc bridging if it goes between marked points on different boundaries. With this terminology we have the following theorem:

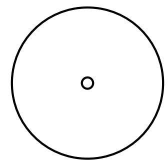
Theorem 6.2. Periodic infinite Frieze patterns of positive integers are in bijection with triangulations of annuli and once-punctured discs.

The bijection is the same in that the marked points correspond to entries of the quiddity row with entries the number of incident triangles.

Example 6.3. The degenerate case is where the infinite periodic quiddity row

$$\cdots \quad 2 \quad 2 \quad 2 \quad \cdots$$

corresponds to the punctured disk below:



There are also cutting and gluing procedures for Frieze patterns. We can cut a 1 from the periodic quiddity sequence $(a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_n)$ by the following:

$$(a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_n) \to (a_1, \ldots, a_{i-1} - 1, a_{i+1} - 1, \ldots, a_n).$$

This procedure give a new Frieze pattern. Gluing is the inverse operation to cutting:

$$((a_1, \ldots, a_n), (b_1, \ldots, b_n)) \to (a_1, \ldots, a_{n-1}, a_{n+1}, 1, b_1 + 1, b_2, \ldots, b_n).$$

We also have another important result:

Theorem 6.3. Given an infinite Frieze pattern with period n, the value

$$m_{i,i+kn+1} - m_{i+1,i+kn}$$

is constant for each $k \geq 1$.

This means that the difference between row kn and row $kn - 2$ is constant. This induces a sequence $\{s_k\}$ of growth coefficients for each infinite Frieze patterns. We have a result regarding this sequence:

Theorem 6.4. The growth coefficients $\{s_k\}$ of a Frieze pattern satisfy

$$s_{k+1} = s_1 s_k - s_{k-1}$$

where $s_0 = 2$.

There is another result which is of more importance as it can determine a lot about the Frieze pattern:

Theorem 6.5. For any infinite Frieze pattern with n-periodic rows and all $1 \leq \ell \leq k$, we have

$$m_{i,j+kn} = s_\ell m_{i,j+(k-\ell)n} + m_{j,(k-\ell)n,i+\ell n}$$

where if $a > b$ we have $m_{a,b} = -m_{b,a}$.

The take-away here is that if we know the first n nontrivial rows of an n-periodic Frieze pattern then we know a lot more information. In particular, if these rows are positive then the entire pattern is positive.

We will now change gears and talk about p-angulations. A dissection of a polygon is an arrangement of non-intersecting interior edges which decomposes the polygon into subpolygons. If all of these polygons are p-gons we call this a p-angulation. In particular, a triangulation is a p-angulation for $p = 3$.

Given a dissection, and a vertex adjacent to polygons of size $p_1, \ldots, p_n$ we associated to it the value

$$\sum_{i=1}^{n} \lambda_{p_i}$$

where $\lambda_{p_1} = 2\cos \pi/p_i$. Observe that λ_p is the ratio of the length of the shortest diagonal of a regular p-gon and the length of a side. We have another result:

Theorem 6.6. Each polygon dissection of an n-gon produces a Frieze pattern of width $n+1$ with entries in the ring of algebraic integers of the field $\mathbb{Q}(\lambda_{p_1}, \ldots, \lambda_{p_s})$.

Specializing to p-angulations the situation simplifies:

Theorem 6.7. There is a bijection between p-angulations of an n-gon and Frieze patterns of width $n+1$ whose quiddity row consists of multiples of λ_p.

The Frieze patterns in this theorem are said to be of type Λ_p. In general, the definition for Frieze patterns of type $\Lambda_{p_1,\ldots,p_s}$ is clear.

6.2 Exercises

1. Let $m_{i,i+1} = 1$ and $m_{i,i} = 0$ for all i. Show that for $j \geq i+2$, $m_{i,j} = m_{i,i+2}m_{i+1,j} - m_{i+2,j}$.

2. Deduce that $m_{i,j}$ is given by the determinant of the matrix

$$\begin{pmatrix} m_{i,i+2} & 1 & & & & & \\ 1 & m_{i+1,i+3} & 1 & & & & \\ & 1 & m_{i+2,i+4} & 1 & & & \\ & & & \ddots & & & \\ & & & 1 & m_{j-4,j-2} & 1 & \\ & & & & 1 & m_{j-3,j-1} & 1 \\ & & & & & 1 & m_{j-2,j} \end{pmatrix}$$

3. Find and prove which annuli $A_{n,m}$ can be p-angulated. Verify that if we set $m = 0$, we get the correct result for S_n.

6.3 The Problem

There are a suite of problems:

1. Does every finite Frieze pattern of type $\Lambda_{p_1,\ldots,p_s}$ come from a dissected polygon? If not, describe the subset of such Frieze patterns which do arise from dissected polygons.

2. Prove or disprove that there is a bijection between p-angulations of S_n and $A_{n,m}$ and period infinite Frieze patterns of type Λ_p.

3. Investigate infinite Frieze patterns from dissections of S_n and $A_{n,m}$. Can the planar ones be characterized? When is there ambiguity?

7 Alcove Walks

7.1 Theory

A Lie group is a group that is also a smooth manifold. Most Lie groups are matrix group and this can be made precise.

Example 7.1. Some common Lie groups are GL_n, SL_n, SO_n, and Sp_n over either $\mathbb{R}$ or $\mathbb{C}$.

Much of the structure of a Lie groups holds over any field (this is somewhat of a miracle), and in general we call these groups Chevalley groups.

Let $G = \mathrm{SL}_n$ over an arbitrary field and let B be the subgroup of G consisting of upper triangular matrices. This is the Boreal subgroup of G.

Example 7.2. If $G = \mathrm{SL}_3$ over $\mathbb{C}$, then B is the set

$$\begin{pmatrix} \mathbb{C} & \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C} & \mathbb{C} \\ 0 & 0 & \mathbb{C} \end{pmatrix}.$$

We call the quotient $V = G/B$ a flag variety. A flag is a sequence of subspaces

$$\{\mathbf{0}\} = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V.$$

where $\dim(V_i) = i$, but we will not discuss this point further. Observe that B is not a normal subgroup so V is not a group, but only a set of cosets. We can resolve this issue if we consider double cosets. So consider the double coset space $B\backslash G/B$ which is the set of sets of the form

$$BgB = \{g' \in G \mid g' = b_1 g b_2 \text{ where } b_1, b_2 \in B\}.$$

Then G admits a Bruhat decomposition

$$G = \bigsqcup_{w \in W} BwB$$

where W is some set of representatives and the sum is disjoint. It turns out that W is a group and we call it the Weyl group of G.

Example 7.3. If $G = \mathrm{SL}_n$ over $\mathbb{C}$ or $\mathbb{R}$, then $W = S_n$.

With this decomposition we have

$$G/B = \bigsqcup_{w \in W} BwB/B$$

The upshot here is that every element gB of G/B corresponds to a unique $w \in W$ and there is a nonunique $b \in B$ such that $gB = bwB$. So we can think of BwB/B as a left coset.

We will now specialize our field. From now on let $G = \mathrm{SL}_n(F)$ where $F = \mathbb{C}((t))$ so that F is the fraction field of $\mathcal{O} = \mathbb{C}[[t]]$. It is well-known that $\mathcal{O}$ admits a maximal ideal (t) and there is a map $\mathcal{O} \to \mathbb{C}$ given by setting $t = 0$. This induces a map $\phi : \mathrm{SL}_n(\mathcal{O}) \to \mathrm{SL}_n(\mathbb{C})$ by setting $t = 0$ in the entries in the matrix. The Iwahori subgroup I is defined by

$$I = \{g \in \mathrm{SL}_n(\mathcal{O}) \mid \phi(g) \in B\}.$$

In other words, it is the inverse image of B under ϕ. We can also describe I as the set

$$\begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ (t) & \mathcal{O} & \mathcal{O} \\ (t) & (t) & \mathcal{O} \end{pmatrix}.$$

We call the quotient G/I an affine flag variety. Again, it is not a group but it admits an Iwahori decomposition

$$G = \bigsqcup_{w \in \tilde{W}} IwI$$

where $\tilde{W}$ is a group called the affine Weyl group.

Example 7.4. Let $G = \mathrm{SL}_3(F)$, and set

$$g = \begin{pmatrix} \frac{1}{t} & 2t & 2t^2 \\ 0 & t & t^2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $g \in B$, and we can decompose g as

$$g = \begin{pmatrix} \frac{1}{t} & 2t & 2t^2 \\ 0 & t & t^2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

the matrix product lives in $B1B$. We can also decompose g as

$$g = \begin{pmatrix} \frac{1}{t} & 2t & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

where the matrix product is again in $B1B$. Now $g \notin I$, but g decomposes as

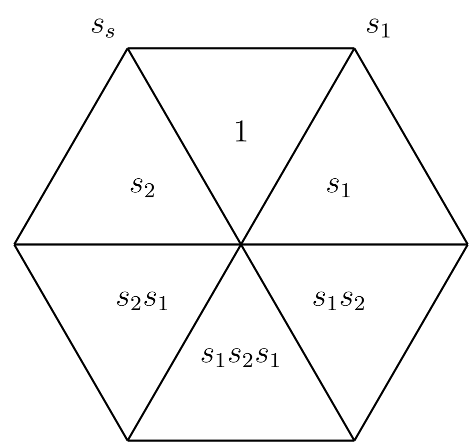
$$g = \begin{pmatrix} 1 & 2 & 2t^2 \\ 0 & 1 & t^2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

where the matrix product is in $I\tilde{W}I$.

Now lets discuss W and $\tilde{W}$ a little more. If $G = \mathrm{SL}_3(F)$, then it turns out that $W = S_3$ and $\tilde{W} = \tilde{S}_3$ where $\tilde{S}_3$ is the affine symmetric group on 3 letters that will be defined in the following. The Coxeter presentation for S_3 is

$$S_3 = \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1, s_1 s_2 s_1 = s_2 s_1 s_2 \rangle$$

where $s_1 = (12)$, $s_2 = (23)$, and $s_1 s_2 s_1 = s_2 s_1 s_2$ is a braid relation. Pictorially we can view this presentation as a triangulated hexagon



described as follows: we call the triangles alcoves and the triangle containing a 1 the fundamental alcove. We label the two interior boundary edges of the fundamental alcove by s_1 and s_2 respectively. Reflecting this fundamental alcove around the hexagon labels every other interior edge with a s_1 or s_2 and as we move from the fundamental alcove into a new alcove we multiply on the left by the corresponding label. By the braid relation the bottom alcove is $s_1s_2s_1 = s_2s_1s_2$. In this sense, elements of S_3 are in bijective correspondence with alcoves in the diagram above.

The Coxeter presentation for $\tilde{S}_3$ is

$$\tilde{S}_3 = \left\langle s_0, s_1, s_2 \mid s_0^2 = s_1^2 = s_2^2 = 1, \substack{s_0s_1s_0=s_1s_0s_1 \\ s_0s_2s_0=s_2s_0s_2 \\ s_1s_2s_1=s_2s_1s_2} \right\rangle .$$

and we can form a similar picture but it is now an infinite tessellation of the plane by hexagons. In particular, $\tilde{S}_3$ is infinite and is in bijective correspondence with alcoves in this infinite setting.

We would now like to find more nuanced information. We will do this by using Steinberg generators. Let U^- be the subgroup of unipotent lower triangular matrices. Then G admits a decomposition

$$G = \bigsqcup_{w \in \tilde{W}} U^- w I$$

where again $\tilde{W}$ is the affine Weyl group. The Steinberg generators are defined to be the matrices

$$x_{\alpha_1}(c) := \begin{pmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad x_{-\alpha_1}(c) := \begin{pmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$x_{\alpha_2}(c) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \quad x_{-\alpha_2}(c) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{pmatrix},$$
$$x_{\alpha_0}(c) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ ct & 0 & 1 \end{pmatrix} \quad x_{-\alpha_0}(c) := \begin{pmatrix} 1 & 0 & ct^{-1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We will write $x_1(c)$, $x_2(c)$, and $x_0(c)$ for $x_{\alpha_1}(c)$, $x_{\alpha_2}(c)$, and $x_{\alpha_0}(c)$ respectively.

Define $n_i(c)$, n_i, and $h_i(c)$ by

$$n_i(c) := x_i(c)x_{-\alpha_i}(-c^{-1})x_i(c), \quad n_i := n_i(1), \quad \text{and} \quad h_i(c) := n_i(c)n_i^{-1}.$$

It's well-know that

$$U^- vI = \{x_{\gamma_1}(d_1) \cdots x_{\gamma_k}(d_k) n_{j_1}^{-1} \cdots n_{j_k}^{-1} I \mid d_1, \dots, d_k \in \mathbb{C}\}.$$

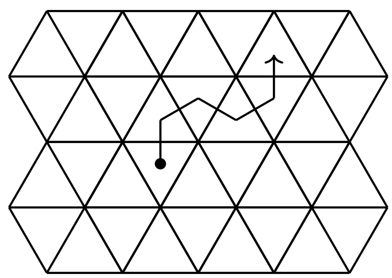
There is also a nice theorem:

Theorem 7.1 (Parkinson-Ram-Schwer)**.** Let $w = s_{i_1} \cdots s_{i_\ell} \in \tilde{W}$ be a reduced expression. Then in G/I we have

$$IwI = \{x_{i_1}(c_1)n_{i_1}^{-1} \cdots x_{i_\ell}(c_\ell)n_{i_\ell}^{-1}I \mid c_1, \ldots, c_\ell \in \mathbb{C}\}.$$

Consider the tessellated plane corresponding to $\tilde{S}_3$. A alcove walk from v to w is a path is this tessellated plane from alcove v to alcove w. We say the walk is minimal if it a shortest path from v to w.

Example 7.5. The following diagram is an example of a minimal alcove walk.



A labeled alcove walk to w is a minimal alcove walk to w where every edge passed in the walk is labelled by an element of $\mathbb{C}$. We then have a theorem:

Theorem 7.2. The elements IwI/I are in bijective correspondence with labelled alcove walks from 1 to w.

There is also a notion of positively folded alcove walks of type w ending in v. For an deep treatment of these types of alcove walks see arXiv:0801.0709. We also hve another bijection:

Theorem 7.3. In G/I, there is a bijection between $(U^- vI \cap IwI)/I$ and labelled positively folded alcove walks of type w ending in v.

7.2 Exercises

1.

(a) Write out all the elements of S_3 as minimal length products of s_1 and s_2. Determine what is special about (13).

(b) Prove that S_3 is in bijection with the alcoves in the first diagram. Use the relations to show this bijection.

(c) Prove that $\tilde{S}_3$ is in bijection with alcoves in the second diagram using the relations as with S_3.

Proof.

(a) The minimal length products are

$$\begin{aligned}(12) &= s_1 \\ (23) &= s_2 \\ (1)(2)(3) &= s_1^2 = s_2^2 \\ (132) &= s_1 s_2 \\ (123) &= s_2 s_1 \\ (13) &= s_1 s_2 s_1 = s_2 s_1 s_2\end{aligned}$$

Observe (13) is the only element that can be written as a nontrivial minimal length product in two different ways (i.e., we exclude (1)(2)(3)).

(b) With the initial labeling of the edges s_1 and s_2, reflecting the fundamental alcove around the hexagon gives induces a unique labeling on each edge. Define an action of S_3 on the fundamental domain as follows: given an element of S_3 move across edges corresponding to that element of S_3 read from left to right. The action is well-defined by the braid relation $s_1 s_2 s_1 = s_2 s_1 s_2$. The action then naturally extends to all alcoves of the hexagon. This action is faithful for otherwise two of the minimal length products in part (a) would be equal which is impossible. It follows that the alcoves of the hexagon are in bijection with elements of S_3

(c) First observe that $\langle s_0, s_1\rangle$, $\langle s_0, s_2\rangle$, and $\langle s_1, s_2\rangle$ are all isomorphic to S_3. We can define an action of $\tilde{S}_3$ on the plane tessellated by hexagons analogous to part (b). The action is well-defined by the braid relations of $\tilde{S}_3$. All that is left to prove is that the action is faithful. If not, then two different alcoves a_1 and a_2 correspond to the same element of $\tilde{S}_3$. They must not belong in the same hexagon by part (b). It follows by (b) again that hexagons containing alcoves a_1 and a_2 have the same labeling. Therefore there exists a hexagon with nondistinct entries contradicting (b).

□

2.

(a) Show that $x_i(c_1)x_i(c_2) = x_i(c_1 + c_2)$ for $i = 0, 1, 2$.

(b) Compute $n_i(c)$ and $h_i(c)$ for $i = 0, 1, 2$. Which of the $x_\alpha(c)$, $n_i(c)$, and $h_i(c)$ are in U^-? Which are in I?

(c) Prove that up to signs, n_0, n_1, and n_2 satisfy the same relations as s_0, s_1, and s_2.

(d) Find and prove an exchange relation for $n_i^{-1}x_j(c)$ for $i, j = 0, 1, 2$.

(e) Prove symbolically that if $c \neq 0$, then

$$x_i(c)n_i^{-1} = x_{-\alpha_i}(c^{-1})x_i(-c)h_i(c).$$

(f) Use parts (d) and (e) to show that when $i \neq j$, we have

$$n_j^{-1}x_i(c)n_i^{-1} \in U^- n_j^{-1} I.$$

Proof.

(a) This follows from matrix multiplication. For example the case $i = 1$ is:

$$x_1(c_1)x_1(c_2) = \begin{pmatrix} 1 & c_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & c_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & c_1 + c_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = x_1(c_1+c_2).$$

The other cases are similar. As a remark, it follows that $x_i(c)^{-1} = x_i(-c)$ by setting $c_1 + c_2 = 0$, and similarly for the $x_{-\alpha_i}$.

(b) We have

$$n_1(c) = \begin{pmatrix} 0 & c & 0 \\ -c^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad n_2(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & c \\ 0 & -c^{-1} & 0 \end{pmatrix}, \quad n_0(c) = \begin{pmatrix} 0 & 0 & -(ct)^{-1} \\ 0 & 1 & 0 \\ ct & 0 & 0 \end{pmatrix},$$

$$h_1(c) = \begin{pmatrix} c & 0 & 0 \\ 0 & c^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad h_2(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c^{-1} \end{pmatrix}, \quad h_0(c) = \begin{pmatrix} c^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{pmatrix}.$$

Clearly the matrices $x_0(c)$, $x_{-\alpha_1}(c)$, $x_{-\alpha_2}(c)$ are in U^-, and the $h_i(c)$ are in U^- exactly when $c = 1$. The $x_i(c)$ and $h_i(c)$ are the only matrices in I unless $c = 0$ in which case the $x_\alpha(c)$, $n_i(c)$, and $h_i(c)$ are all in I.

(c) The matrices n_i are

$$n_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad n_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \text{and} \quad n_0 = \begin{pmatrix} 0 & 0 & -t^{-1} \\ 0 & 1 & 0 \\ t & 0 & 0 \end{pmatrix}.$$

Up to a change of sign in the entries (i.e., changing the -1s to 1s), the matrices are all of order 2. It can be checked directly that they satisfy the braid relations

$$n_i n_j n_i = n_j n_i n_j.$$

This implies the n_i satisfy the same relations as s_i up to sign.

(d) We claim

$$n_i^{-1} x_j(c) = x_{s_i(\alpha_j)}(\pm c) n_i^{-1}.$$

where $s_i(\alpha_j)$ is the action of the affine symmetric group on the affine Weyl group. The precise calculations for the sign can be quickly checked. As a remark, this action extends to the $-\alpha_j$ trivially so the exchange relation holds for the $x_{-\alpha_j}(c)$ as well.

(e) Suppose $c \neq 0$ Then we have the following series of equivalent statements:

$$\begin{aligned} x_i(c) n_i^{-1} &= x_{-\alpha_i}(c^{-1}) x_i(-c) h_i(c) \\ x_i(c) n_i^{-1} h_i(c)^{-1} &= x_{-\alpha_i}(c^{-1}) x_i(-c) \\ x_i(c) n_i^{-1} n_i n_i(c)^{-1} &= x_{-\alpha_i}(c^{-1}) x_i(-c) \\ x_i(c) n_i(c)^{-1} &= x_{-\alpha_i}(c^{-1}) x_i(-c) \\ x_i(c) x_i(c)^{-1} x_{-\alpha_i}(-c^{-1})^{-1} x_i(c)^{-1} &= x_{-\alpha_i}(c^{-1}) x_i(-c) \\ x_{-\alpha_i}(-c^{-1})^{-1} x_i(c)^{-1} &= x_{-\alpha_i}(c^{-1}) x_i(-c) \end{aligned}$$

The last statement is indeed equality by the remark in part (a).

(f) Using part (e) we rewrite $n_j^{-1} x_i(c) n_i^{-1}$ as

$$n_j^{-1} x_{-\alpha_i}(c^{-1}) x_i(-c) h_i(c),$$

and applying the exchange relation in part (d) with $-\alpha_i$ yields

$$x_{s_j(-\alpha_i)}(\pm c^{-1}) n_j^{-1} x_i(-c) h_i(c).$$

By part (a) we know $x_i(-c) h_i(c) \in I$, and it can be easily checked that $x_{s_j(-\alpha_i)}(\pm c^{-1}) \in U^-$ when $i \neq j$.

□

3.

(a) How many alcove walks of type w are there?

(b) Describe the elements of IwI.

(c) How many positively folded alcoves walks of type w ending in v are there?

(d) Describe the elements of $U^{-1}vI \cap IwI$.

7.3 The Problem

There are a suite of problems:

1. For $G = \mathrm{SL}_3$ and given $w, v_1, v_2 \in \tilde{W}$, when is $(U^- v_1 I) \cap (IwI) \cap (U^+ v_2 I)$ nonempty? Find a combinatorial formula for its measure.

2. Can we find similar results for other Chevalley groups?

3. Can we use results on triple intersections to compute certain integrals on G?

8 Virtual Resolutions

8.1 Theory

A monomial ideal is an ideal that can be generated by monomials.

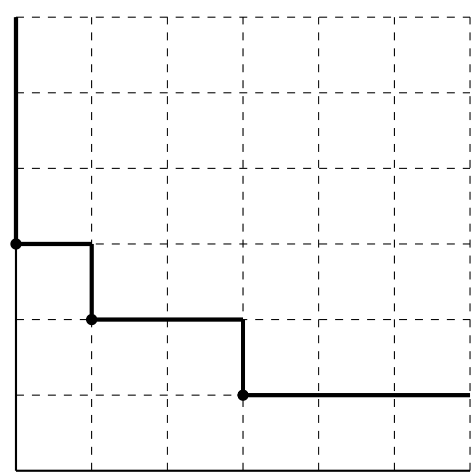
Example 8.1. The ideal $\langle x-y, y\rangle$ is a monomial ideal because it is the same as the ideal $\langle x, y\rangle$. The ideal $\langle x^2, y^2, z^2\rangle$ is clearly a monomial ideal.

Staircase diagrams are a pictorial way to characterize monomial ideals in two variables. They rely on the following facts:

- A monomial ideal is uniquely characterized by the set of monomials it contains.
- Every monomial ideal has a unique minimal set of monomial generators.

They are a graph in the (x, y)-plane where the (i, j)-entry corresponds to $x^i y^j$. To draw the staircase diagram corresponding to a monomial ideal I, first write it as an ideal generated by minimal set of monomial generators. Then mark the points corresponding to the generators and draw a staircase between them from the top left to bottom right. This is best illustrated with an example:

Example 8.2. Let $I = \langle x^3y, xy^2, y^3\rangle$. The staircase diagram corresponding to I is

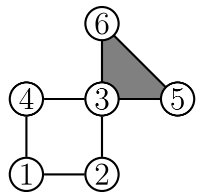


An important fact is that the monomials corresponding to the region above the staircase are all contained in the monomial ideal. We call a monomial ideal squarefree if its generators contain no squared factors. In other words, none of the variables show up in a generator degree larger than two. In particular, all marked point in the staircase diagram occur in the 2×2 square.

Example 8.3. The ideals $\langle x, y\rangle$ and $\langle x^2, y^2, z^2\rangle$ are monomial ideals

We will now change gears a little and discuss simplicial complexes and random graphs. For a simplicial complex Δ on n vertices, define the Stanley-Reisner ideal $I_\Delta \subset k[x_1, \dots, x_n]$ to be the ideal generated by the minimal non-faces.

Example 8.4. Consider the simplicial complex Δ given by



The minimal non-faces are all the missing edges in the simplicial complex, so

$$I_\Delta = \langle x_1x_3, x_1x_5, x_1x_6, x_2x_4, x_2x_5, x_2x_6, x_4x_5, x_4x_6\rangle.$$

We then have a theorem which essentially lets us compute the Betti numbers easily:

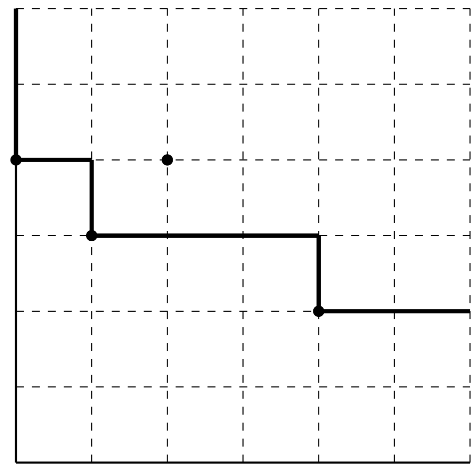
Theorem 8.1 (Hochster's Formula)**.** For a simplicial complex Δ we have

$$\beta_{i,j}(S/I_\Delta) = \sum_{|\alpha|=j} \dim(\tilde{H}_{i-j-1}(\Delta|_\alpha)).$$

We would now like to construct a model for a "random monomial ideal". The inspiration for this comes from random graph theory and the following theorem:

Theorem 8.2. Choose a random graph G with $M(n)$ edges on n vertices uniformly. Then for $\epsilon > 0$ as $n \to \infty$ if $M(n) \geq (1+\epsilon)n \log n$, then asymptotically almost surely, the graph is connected. Conversely, if $M(n) < (1-\epsilon)n \log n$, then asymptotically almost surely the graph is disconnected.

In our setting, let n be the number of variables, D be the maximum degree, and p be the probability of taking a particular monomial. If $n = 2$, $D = 6$, and $p = 0.1$, then an example of a randomly generated monomial with these parameters is represented by the staricase diagram



Observe that one of the monomials is redundant as it lies above the staircase.

We can also describe another probability model for random simplicial complexes as follows: let n be the number of variables (i.e., the number of vertices) and p be the attaching probability (i.e., the probability that any two vertices have an edge between them). Choosing such a random graph we then form the largest simplicial complex from it. With this model we have a theorem:

Theorem 8.3. Fix some $r \geq 1$. Let $\Delta \sim \Delta(n,p)$ with $1/n^{1/r} << p << 1/n^{2/(2r+1)}$, then asymptotically almost surely $r+1 \leq \operatorname{reg}(S/I_\Delta) \leq 2r$.

Now having the necessary background material, the goal is to understand syzygies over a product of projective spaces. For $\mathbf{n} \in \mathbb{N}^r$ we will write $P^{\mathbf{n}}$ for $P^{n_1} \times \cdots \times P^{n_r}$. We say the polynomial ring $k[x_1, \ldots, x_n]$ is $\mathbb{Z}^r$-graded if $\deg(x_i)$ is an element of $\mathbb{Z}^r$. We call a polynomial f in a $\mathbb{Z}^r$-graded polynomial ring homogeneous if the degree in every term is the same.

Example 8.5.

- The polynomial ring $k[x_1, \ldots, x_n]$ with the standard grading is $\mathbb{Z}$-graded with $\deg(x_i) = 1$.
- Consider the polynomial ring $k[x_0, x_1, y_0, y_1, y_2]$ with $\deg(x_i) = (1,0)$ and $\deg(y_i) = (0,1)$. Then $\deg(x_0x_1) = (2,0)$ and $\deg(x_1^2y_1y_2) = (2,2)$. This grading is the homogeneous coordinate ring for the space $\mathbb{P}^1 \times \mathbb{P}^2$.

The $\mathbb{P}^{\mathbf{n}}$ homogeneous coordinate ring S is

$$S := k[x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{r,0}, \ldots, x_{r,n_r}]$$

with $\deg(x_{i,j}) = e_i$ where e_i is the i-th standard basis vector in $\mathbb{Z}^r$. The irrelevant ideal B is defined by

$$B := \bigcap_{i=1}^{r} \langle x_{i,0}, \ldots, x_{i,n_i} \rangle.$$

The irrelevant ideal B corresponds to the coordinates that don't have any geometric realization in $\mathbb{P}^n$. That is, it corresponds to the "invalid coordinates".

Example 8.6. For $\mathbb{P}^1 \times P^2$, the irrelevant ideal is $B = \langle x_0, x_1 \rangle \cap \langle y_0, y_1, y_2 \rangle$. If $f \in B$, then f is zero on all the coordinates $([a_0 : a_1], [b_0 : b_1 : b_2])$ where either $a_1 = a_0 = 0$ or $b_0 = b_1 = b_2 = 0$.

The saturation of an ideal I by an ideal B (B will stand for the irrelevant ideal everywhere but this definition) is given by

$$I : B^\infty = \langle r \in S \mid rB^k \subset I \text{ for } k \text{ sufficiently large} \rangle.$$

Geometrically, the saturation removes the component corresponding to B. For varieties a similar situation holds: $V(I : B^\infty) = \overline{V(I) - V(B)}$.

Example 8.7. Let $I = \langle x_0^2, x_0 y_0, x_1 y_0 \rangle$ and $B = \langle x_0 y_0, x_0 y_1, x_1 y_0, x_1 y_1 \rangle$. Then

$$I : B^\infty = \langle x_0^2, y_0 \rangle.$$

In fact, we have a correspondence:

Proposition 8.1. Subvarieties of a product of projective spaces correspond to homogeneous B-saturated radial ideals in the homogeneous coordinate ring.

We will now discuss virtual resolutions. A complex

$$C : C_0 \xleftarrow{d_0} C_1 \xleftarrow{d_1} C_2 \xleftarrow{d_2} \cdots$$

is called a virtual resolution if the C_i are free modules, $d_i \circ d_{i-1} = 0$ for $i \geq 0$, and $H_i(C_*)$ is irrelevant for $i > 0$. We use virtual resolutions because over $\mathbb{P}^{\mathbf{n}}$ minimal free resolutions don't accurately reflect the geometry. We have a well-known theorem for $\mathbb{P}^n$:

Theorem 8.4. If I is a nonmaximal $\mathbb{Z}^r$-graded ideal on $\mathbb{P}^n$, then S/I has a free resolution of length at most n

There is a corresponding theorem for $\mathbb{P}^{\mathbf{n}}$:

Theorem 8.5. Every finitely generated $\mathbb{Z}^r$-graded B-saturated module on $\mathbb{P}^{\mathbf{n}}$ has a virtual resolution of length at most $n_1 + \cdots + n_r$.

We also have a particularly nice lemma:

Lemma 8.1. If I is a B-saturated ideal, and $J : B^\infty = I$, then a minimal free resolution ofS/J is a virtual resolution of S/I.

In our setting there is a notion of multigraded regularity, see arXiv:math/0305214.

8.2 Exercises

1.

(a) Prove that a monomial ideal is uniquely characterized by the set of monomials it contains.

(b) Prove that every monomial ideal has a unique minimal set of monomial generators.

Proof.

(a) Suppose I and J are two different monomial ideals containing the same set of monomials. Since they are monomial ideals, their monomials form a set of generators for the ideal. But then $I = J$.

(b) Let I be a monomial ideal and suppose we have two different minimal generating sets $\{x^{\alpha_1}, \dots, x^{\alpha_n}\}$ and $\{x^{\beta_1}, \dots, x^{\beta_n}\}$. Then x^{α_i} is a linear combination of the x^{β_j}, but x^{α_i} is a monomial so $x^{\alpha_i} = x^{\beta_j}$ for some j. Since i and j were arbitrary this proves the generating set is unique in the case of finitely generated ideals. The infinite case is proven in exactly the same way.

□

2.

(a) Given the monomial ideal $\langle x_0x_1^2y_0, y_0y_1^2\rangle$, compute its saturation with respect to $\langle x_0y_0, x_0y_1, x_1y_0, x_1y_1\rangle$. You may assume that the saturation of a monomial ideal is a monomial ideal.

(b) Check this using Macaulay 2.

Proof.

(a) We are working in $S = k[x_0, x_1, y_0, y_1]$. Declare ideals $I = \langle x_0x_1^2y_0, y_0y_1^2\rangle$ and $B = \langle x_0y_0, x_0y_1, x_1y_0, x_1y_1\rangle$. We need to compute

$$I : B^\infty = \langle r \in S \mid rB^k \subset I \text{ for } k \text{ sufficiently large}\rangle.$$

We claim $I : B^\infty = I$. The containment $I \subseteq I : B^\infty$ is trivial. For the reverse containment suppose $r \in I : B^\infty$. Then $rB^k \subset I$ for sufficiently large k. Since no products of the generators of B are multiples of the generators of I, this implies $r \in I$.

(b) This is easily written in Macaulay 2.

□

3.

(a) Use Macaulay 2 to compute some examples of multigraded regularity.

(b) Write code to compute the resolution regularity.

8.3 The Problem

The problem is to use random methods to characterize the virtual resolutions of monomial ideals that are given by free resolutions of monomial ideals. In particular we would like to answer the following:

1. Which multidegrees show up as twists in virtual resolutions?
2. What can we say about the "virtual resolution regularities", do they still give bounds on the multigraded regularity?
3. Is there any structure to the set of virtual resolutions coming from monomial ideals?
4. Can we answer any of the above questions for monomial modules?

9 q-Rationals

9.1 Theory

For each $n \in \mathbb{N}$, define the polynomial $[n]_q \in \mathbb{Z}[q]$ by

$$[n]_q := 1 + q + q^2 + \cdots + q^{n-1}.$$

As a remark, substituting $q = 1$ gives n. We would like to extend this q-analogue to rational numbers

$$\left[\frac{r}{s}\right]_q$$

. There are multiple ways to do this, but we want this q-analogue of rational numbers to satisfy ordering and convergence properties analogous to that of the rational numbers. That is, we want the following:

- A partial ordering on $\left[\frac{r}{s}\right]_q$ that restricts to the partial ordering on $\mathbb{Q}$ when $q = 1$.
- If $\frac{r}{s} \to \lambda$ with λ irrational, then we want $\left[\frac{r}{s}\right]_q$ to converge to some power series such that we obtain λ when $q = 1$.

A naive guess would be

$$\left[\frac{r}{s}\right]_q := \frac{[r]_q}{[s]_q} = \frac{1 + q + \cdots + q^{a-1}}{1 + q + \cdots + q^{b-1}},$$

but this does not satisfy the desirable properties above.

We say a continued fraction is an expression of nested fractions:

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\cdots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}.$$

We denote the above expression by $[a_1, a_2, \ldots, a_n]$.

Example 9.1. Observe

$$\frac{7}{4} = 1 + \frac{1}{1 + \frac{1}{3}},$$

and the continued fraction expansion is $[1, 1, 3]$.

The expression $[a_1, a_2, \dots, a_n]$ is not unique but requiring an even or odd number of coefficients makes it unique.

Example 9.2. We can write $[1, 1, 2, 1]$ for $\frac{7}{4}$ as well as $[1, 1, 3]$.

The proper definition for q-rational numbers is as follows: if $\frac{r}{s} = [a_1, a_2, \dots, a_{2n}]$, then

$$\left[\frac{r}{s}\right]_q := [a_1]_q + \cfrac{q^{a_1}}{[a_2]_{q^{-1}} + \cfrac{q^{-a_2}}{\dots + \cfrac{q^{a_{2n-1}}}{[a_{2n}]_{q^{-1}}}}}.$$

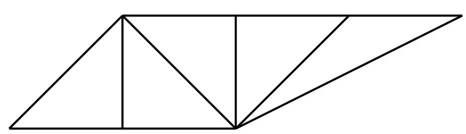
Example 9.3. Write $\frac{7}{3} = [2, 3]$. Then

$$\left[\frac{7}{3}\right]_q = (1+q)\frac{q^2}{1+q^{-1}+q^{-2}} = \frac{1+2q+2q^2+q^3+q^4}{1+q+q^2}.$$

The only time the naive guess agrees with the definition is for $\left[\frac{n+1}{n}\right]_q$. It turns out that this definition for q-rational numbers satisfies the ordering property.

There is also a combinatorial interpretation for q-rational numbers. Given $\frac{r}{s} = [a_1, a_2, \dots, a_n]$, construct a triangulated polygon $T_{r/s}$ as follows: draw a_1 adjacent triangles with their bases facing down all incident to a common vertex, then draw a_2 adjacent triangles with their bases facing up all incident to a common vertex such that the last triangle coming from a_1 and the first triangle coming from a_2 share a common edge, and repeat this procedure for $a_3, \dots, a_n$.

Example 9.4. For $\frac{7}{3} = [2, 3]$, $T_{7/3}$ is

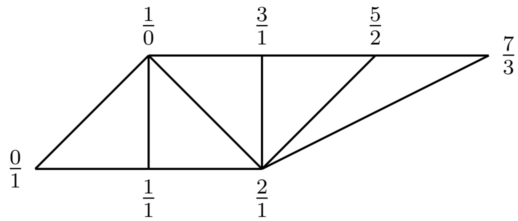


We will put a labeling on $T_{r/s}$. To do this, we first define the Farey sum of two rational numbers by

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}.$$

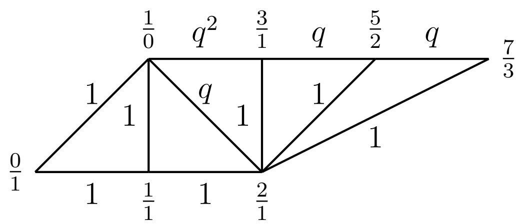
Start by labeling the left two vertices of $T_{r/s}$ by $\frac{0}{1}$ and $\frac{1}{0}$. Proceeding from left to right, for each triangle, label the third vertex as the Farey sum of the previous two.

Example 9.5. For $\frac{7}{3} = [2, 3]$, $T_{7/3}$ is



For the top vertices, from left to right, we label edges incident to that vertex counter-clockwise with increasing powers of q starting at 1 and ignoring the first unlabeled edge. All other edges are labeled with 1.

Example 9.6. For $\frac{7}{3} = [2, 3]$, $T_{7/3}$ is

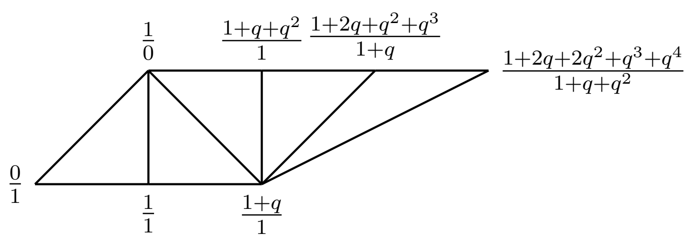


Observe that in each triangle, two edges incident to the third vertex are weighted with at 1 and q_k for some k. If we instead labeled the vertices from left to right using the weighted Farey sum rule

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{a + q^k c}{b + q^k d}$$

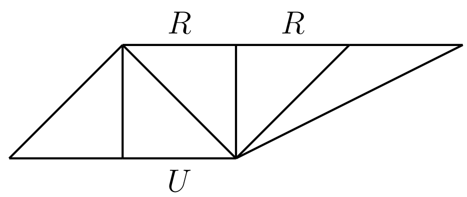
where vertex $\frac{c}{d}$ is incident to the edge in the triangle with weight q^k, then the right most vertex of $T_{r/s}$ will be labeled with $\left[\frac{r}{s}\right]_q$. This gives a geometric method to compute q-rational numbers.

Example 9.7. The triangulation $T_{7/3}$ with weighted Farey vertices is, and omitted edge labels, is



We also have another interpretation. From the triangulation $T_{r/s}$ construct a binary word in the alphabet $\{R, U\}$ as follows: ignore the first and last triangles. For the others, label their boundary edges U if their bases are facing down, and R up their bases are facing up.

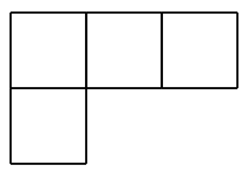
Example 9.8. For $\frac{7}{3} = [2, 3]$, we label



The binary word is URR.

From this binary word we will construct a snake graph $G_{r/s}$ as follows: start with a square. For each letter in the binary word add another square either above (for U) or to the right (for R) of the previous square.

Example 9.9. The snake graph for $G_{7/3}$ is



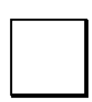
In the literature our "snake graph" is referred to as the "dual snake graph". Let $L(G_{r/s})$ be the set of all lattice paths for the snake graph $G_{r/s}$. Then we have a theorem:

Theorem 9.1. We have

$$|L(G_{r/s})| = r \quad \text{and} \quad L(\hat{G}_{r/s})$$

where $\hat{G}_{r/s}$ is the snake graph for $[a_2, a_3, \ldots, a_n]$.

We can put a partial ordering on the lattice paths in $G_{r/s}$ induced from the partial order defined by making



less than

$$\square$$

In particular, this means there is a minimal path in $G_{r/s}$. We define the height (or rank) of a lattice by how many steps it takes to get to it from the minimal path. We have a theorem:

Theorem 9.2. Let $\left[\frac{r}{s}\right]_q = \frac{R(q)}{S(q)}$. Then the coefficient of q^k in $R(q)$ is the number of lattice paths in $G_{r/s}$ of height k; the coefficient of q^k in $S(q)$ is the number of lattice paths in $\hat{G}_{r/s}$ of height k.

We will quickly discuss convergence of q-rationals. Given an infinite sequence $a_1, a_2, \ldots$, define a sequence of rational numbers x_n by

$$x_n := [a_1, a_2, \ldots, a_n]$$

called convergents. Then the sequence x_n converges to a real number denoted by the infinite continued fraction $[a_1, a_2, \ldots]$.

Example 9.10. Some infinite continued fractions are

- $\sqrt{2} = [1, 2, 2, 2, \ldots]$.
- $\sqrt{3} = [1, 1, 2, 1, 1, 2, \ldots]$.
- $\varphi = [1, 1, 1, 1, \ldots]$.

If $\frac{r}{s} \to \lambda$, then writing $\left[\frac{r}{s}\right]_q$ as a Taylor expansion in q, we say that $\left[\frac{r}{s}\right]_q$ converges if its Taylor coefficients eventually stabilize. Almost nothing is known about these "q-real numbers" or the Taylor coefficients.

9.2 Exercises

1. Find an example where

$$\left[\frac{r}{s}\right]_q = \frac{[r]_q}{[s]_q}$$

does not satisfy the ordering property.

Proof. The idea is to find positive integers a, b, c, and d, such that $ad > bc$ but $b+c > a+d$ for then the leading term in $[a]_q[d]_q - [b]_q[c]_q$ will be negative. Let $a = 3$, $d = 4$, $b = 7$, and $c = 1$. Then $\frac{3}{7} > \frac{1}{4}$, but

$$[3]_q[4]_q - [1]_q[7]_q = (1+q+q^2)(1+q+q^2+q^3) - (1+q+q^2+q^3+q^4+q^5+q^6)$$

has leading term $-q^6$. □

2.

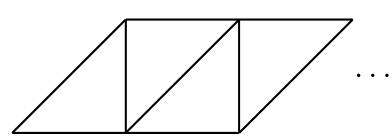
(a) What does $T_{r/s}$ look like for $[1, 1, \ldots, 1]$?

(b) Prove that $[1, 1, \ldots, 1]$ is always a ratio of Fibonacci numbers.

(c) Use the triangulation method to compute $\left[\frac{5}{3}\right]_q$ and $\left[\frac{5}{8}\right]_q$.

Proof.

(a) The triangulation is

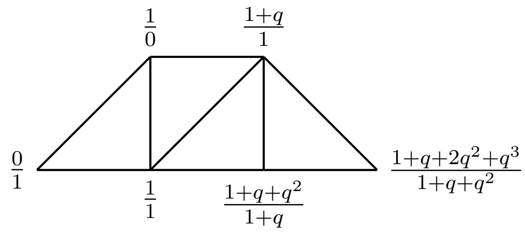


(b) By the Farey sum law, the label of a vertex of the triangulation in (a) will be a ratio of Fibonacci numbers since the starting labels are $\frac{0}{1}$ and $\frac{1}{0}$. The rightmost vertex is the fraction $[1, 1, \ldots, 1]$ so it is also a ration of Fibonacci numbers.

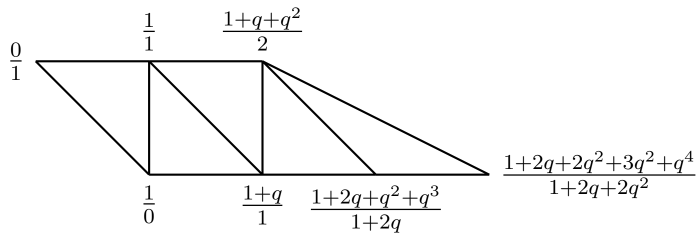
(c) First observe

$$\frac{5}{3} = 1 + \frac{1}{1+\frac{1}{2}} \quad \text{and} \quad \frac{5}{8} = 0 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{2}}}}.$$

Therefore $\frac{5}{3} = [1, 1, 2]$ and $\frac{5}{8} = [0, 1, 1, 1, 2]$. The corresponding triangulations are



and



Therefore

$$\left[\frac{5}{3}\right]_q = \frac{1+q+2q^2+q^3}{1+q+q^2} \quad \text{and} \quad \left[\frac{5}{8}\right]_q = \frac{1+2q+2q^2+3q^2+q^4}{1+2q+2q^2}.$$

□

3. Write down all the lattice paths in $G_{8/5}$ and draw the Hasse diagrams.

9.3 The Problem

There are a suite of problems:

1. Is there a combinatorial description of the L polynomials in the Cluster Algebras section related to the q-rationals?

2. Make progress towards proving that the numerators and denominators in q-rationals are unimodal.

3. Is there a pattern to the Taylor coefficients for "q-real numbers"?

10 Quotes

- "Mathematics is like slamming your head against a wall again and again. The only question is which will crack first, your head or the wall." - Gregg.
- "Graduate students keeping it under their pillow and assimilate it while they sleep." - Ben on Symmetric Functions and Hall Polynomials text.
- "Schur functions are proof that God loves us and wants us to be happy." - Claire.
- "Who says you can't quantify love?" - Andy.
- "It worked once, so maybe it'll work in other cases." - Claire.
- "I did not sanction "super mentor" but I will allow it..." - Ben.
- "I'm not just making this up." - Andy.
- "Because we're mathematicians we're not going to really do it, we're just going to pretend we actually did it." - Andy.
- "That's a topologist's proof: "look at this picture"." - Nick.
- "interesting enough to be nontrivial but easy enough to be tractable." - Nick.
- "Is there nothing special about the special linear group?" - Vijay.
- "The special thing is that it gives us pretty pictures." - Andy.
- "I wonder if there's a way to extend this to generalized dumplings." - Gregg.
- "Evaluating these lattice partition functions is the key to all combinatorics." - Ben.
- "In a flagrant abuse of notation..." - Alan.
- "One person's rational function is another person's clearing denominator." - Ben.
- "...so that might be disappointing if you think about it too hard." -Vijay.
- "You could just use a single hexagon and pac-man-boundary-conditions." - Henry.