Asymptotics of moments of Dirichlet L-series and Kac-Moody Lie algebras

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1 Motivation

In Analytic Number Theory we are interested in understanding the analytic properties of zeta and L-functions, i.e, the Riemann Hypothesis. For example, if

$$\pi(x) := \sum_{\substack{p \text{ prime} \\ p \leq x}} 1$$

then the Riemann hypothesis is equivalent to $\pi(x) = \mathrm{Li}(x) + O(x^{\frac{1}{2}} \log x)$. In general, it suffices to to have strong bounds of the underlying L-functions on the critical line. For example: for small $\epsilon > 0$ and $r \geq 1$, and $T > 1$,

$$\int_1^T |\zeta(\frac{1}{2} + it)|^{2r} \, dt = T P_r(\log T) + O(T^{\frac{1}{2}+\epsilon})$$

where $P_r(\log T)$ is a polynomial in $\log T$ of degree r^2 implies the Lindelöf Hypothesis, and has large applications for the Riemann Hypothesis.

In the 1980's an idea emerged where it could be useful to consider an *averaging family* of L-functions to create a multiple Dirichlet series. L-functions alone behave chaotically, but if a good averaging family is chosen the associated multiple Dirichlet series has nice properties and gives information regarding the analytic properties of the original L-functions in the family. It has yet to be made precise which families are nice, and this is in part why the study of multiple Dirichlet series is hard.

2 Setup

For $d \in \mathbb{Z}$ non-zero and square-free, let

$$\chi_d(n) := \begin{cases} \left(\dfrac{d}{n}\right) & d \equiv 1 \pmod 4 \\ \left(\dfrac{4d}{n}\right) & d \equiv 2, 3 \pmod 4 \end{cases}$$

where $\left(\dfrac{a}{b}\right)$ is the Kronecker symbol defined by

$$\left(\frac{a}{b}\right) := \left(\frac{a}{u}\right) \prod_{i=1}^{k} \left(\frac{a}{p_i}\right)^{e_i}$$

where $b = u \cdot p_1^{e_1} \cdots p_k^{e_k}$ is the prime factorization of b, and $\left(\dfrac{a}{p_i}\right)$ is the Legendre symbol defined by

$$\left(\frac{a}{p_i}\right) := \begin{cases} 1 & a \text{ is a quad. res. modulo } p_i \text{ and } a \not\equiv 0 \pmod{p_i} \\ -1 & a \text{ is a non quad. res. modulo } p_i \\ 0 & a \equiv 0 \pmod{p_i} \end{cases}.$$

χ_d is the quadratic character attached to $\mathbb{Q}(\sqrt{d})$. Define the L-function associated to χ_d by

$$L(s, \chi_d) := \sum_{n \geq 1} \chi_d(n) n^{-s} = \prod_{p \text{ prime}} (1 - \chi_d(p) p^{-s})^{-1} \qquad (\text{for } \Re(s) > 1).$$

These are the L-functions in the number field setting. We have similar L-functions in the function field setting. Indeed, if $\mathbb{F}_q$ is a finite field of odd characteristic, over $\mathbb{F}_q(x)$ we have an analogous character $\chi_d(m) = \left(\dfrac{d}{m}\right)$ for $d, m \in F_q[x]$ monic. The associated L-function is defined by

$$L(s, \chi_d) := \sum_{m \text{ monic}} \chi_d(m) |m|^{-s} = \prod_{\pi \text{ monic irr.}} (1 - \chi_d(\pi) |\pi|^{-s})^{-1} \qquad (\text{for } \Re(s) > 1)$$

where $|a| = q^{\deg a}$.

In the number field setting, for $r \in \mathbb{N}$ and $D > 0$, define the r-th moment by

$$M_r(D) := \sum_{\substack{d \text{ sq.-free} \\ |d| = D}} L(\tfrac{1}{2}, \chi_d)^r.$$

We would like to study the asymptotics of this average. Often it's convenient to work with a smoothed version:

$$M_r(D;F) := \sum_{\substack{d \text{ sq.-free} \\ |d|=D}} L(\tfrac{1}{2}, \chi_d)^r F(\tfrac{d}{D}).$$

for suitably nice functions $F : (0, \infty) \to [0, 1]$; purely analytic methods produced asymptotics for the original moment from the smoothed version. In the function field setting we define the r-th moment as

$$M_r(D) := \sum_{\substack{d \text{ monic sq.-free} \\ |d|=D}} L(\tfrac{1}{2}, \chi_d)^r.$$

3 Conjectures

In 2005, Conrey, Farmer, Keating, Rubinstein, and Snaith put forth the following conjecture for the asymptotics of moments in the number field setting (see [1]):

Conjecture 3.1 (CFKRS)**.** For $r \geq 1$, nice functions F and small $\epsilon > 0$,

$$M_r(D;F) = D\mathfrak{Q}_r^F(\log D) + O(D^{\frac{1}{2}+\epsilon})$$

where $\mathfrak{Q}_r^F(t)$ is a polynomial of degree $\frac{r(r+1)}{2}$.

In 2019, it was shown by Diaconu (see [2]) that additional secondary terms emerge in the asymptotic formula for the CFKRS conjecture. So, coming up with the correct conjecture for the asymptotic behavior is difficult. The refined conjecture for the number field case is:

Conjecture 3.2. For $r \geq 4$, $D \geq 1$, nice F, and small $\epsilon > 0$,

$$M_r(D;F) = \sum_{n=1}^{\infty} D^{\frac{1}{2}+\frac{1}{2n}}\mathfrak{Q}_{n,r}^F(\log D) + O(D^{\frac{1}{2}+\epsilon})$$

for explicitly computable polynomials $\mathfrak{Q}_{n,r}^F(t)$.

The analogous conjecture for the function field setting is:

Conjecture 3.3. For $r \geq 4$, $D \in \mathbb{N}$, and small $\epsilon > 0$,

$$M_r(D) = \frac{q^D}{\zeta(2)}Q_1(D,q) + \sum_{n=2}^{\infty} q^{D(\frac{1}{2}+\frac{1}{2n})}Q_n(D,q) + O_{\epsilon,q,r}(q^{D(\frac{1}{2}+\epsilon)})$$

for explicitly computable $Q_n(D,q)$.

The takeaway from these conjectures is that the right-hand sides no longer contain quadratic Dirichlet L-functions at the expense of error terms.

The above conjectures are also important for the following reason: often there is an analogy between arguments in the function field setting and the number field setting, albeit the function field setting is drastically easier. It has been proved that if the refined conjecture in the function field setting is assumed then function field Riemann follows. We hope the proof of the refined conjecture in the function field setting will have an analogous proof for number fields, and this will provide some insight on how to prove number field Riemann. My research concerns computing the $\mathfrak{Q}_{n,r}^F$ and the Q_n. The current goal is to compute the Q_n because the calculations in the function field setting are easier to compute from a technical standpoint.

4 Computing The Q_n

So how do we compute the Q_n? It suffices to consider a multiple Dirichlet series which is closely connected to the moments for the function field setting (setting all $s_i = \frac{1}{2}$ makes this expression look very close to the definition for moments):

$$Z(s_1, \ldots, s_{r+1}) = \sum_{d \text{ sq.-free}} \frac{L(s_1, \chi_d) \cdots L(s_r, \chi_d)}{|d|^{s_{r+1}}}.$$

The sum in this multiple Dirichlet series runs over the fundamental discriminants which makes its analytic behavior difficult to analyze. Instead, we work with a modified series $Z^*(s_1, \ldots, s_{r+1})$ whose definition we will avoid due to the construction of this modified series being highly nontrivial. Assuming only the meromorphic continuation of $Z^*(s_1, \ldots, s_{r+1})$, it satisfies a group of functional equations

$$\begin{aligned} Z^*(w_i(s_1, \ldots, s_{r+1})) &= Z^*(s_1, \ldots, s_{r+1}) \qquad \text{(for } 1 \leq i \leq r) \\ Z^*(w_{r+1}(s_1, \ldots, s_{r+1})) &= Z^*(s_1, \ldots, s_{r+1}) \end{aligned}$$

where

$$\begin{aligned} w_i(s_1, \ldots, s_{r+1}) &= (\ldots, s_{i-1}, 1 - s_i, s_{i+1}, \ldots, s_{r+1} + s_i - \tfrac{1}{2}) \qquad \text{(for } 1 \leq i \leq r) \\ w_{r+1}(s_1, \ldots, s_{r+1}) &= (s_1 + s_{r+1} - \tfrac{1}{2}, \ldots, s_r + s_{r+1} - \tfrac{1}{2}, 1 - s_{r+1}) \end{aligned}$$

$Z^*(s_1, \ldots, s_{r+1})$ has a simple pole at $(\frac{1}{2}, \ldots, \frac{1}{2}, 1)$. In fact, all the poles of $Z^*(s_1, \ldots, s_{r+1})$ are reflections of this pole by products of the w_i and w_{r+1}. If we can compute all of the poles, standard analytic methods produce analytic information about $Z^*(s_1, \ldots, s_{r+1})$ and hence $Z(s_1, \ldots, s_{r+1})$ as well. We can use this pole data to compute the Q_n explicitly.

5 Kac-Moody Lie Algebras and Root Systems

One major drawback is that the group W_r generated by $w_1, \dots, w_{r+1}$ is infinite for $r \geq 4$, so we need some powerful machinery to compute the poles. This machinery is the theory of Kac-Moody Lie algebras. A Kac-Moody Lie algebra can be characterized by the following:

- A generalized Cartan matrix $C = (c_{ij})$.
- A set of linear independent vectors α_i, called roots, over a complex vector space.
- A set of linear independent vectors $\alpha_i^\vee$, called coroots, in the dual vector space such that $\alpha_i^\vee(\alpha_j) = c_{ij}$.

The roots define a root system: a subset of vectors Φ contained in the $\mathbb{Z}$-span of the α_i which are invariant under a group W of reflections called the Weyl group.

In our setting W_r is isomorphic to the Weyl group of the Kac-Moody Lie algebra with generalized Cartan matrix

$$\begin{pmatrix} 2 & & & -1 \\ & 2 & & \vdots \\ & & \ddots & -1 \\ -1 & \cdots & -1 & 2 \end{pmatrix}$$

Because this generalized Cartan matrix is nice, computing the action of the reflections on the α_i is easy. Computing all reflections of the simple pole is equivalent to finding all real roots of this Kac-Moody Lie algebra; given a real root we can substitute the original pole in to get the reflected pole. We would like a closed formula to compute the real roots. Finding such a closed formula is difficult and this is what I'm currently working to achieve. The difficulty is that the roots associated to this Weyl group are of two types: real and imaginary (imaginary roots do not appear in finite root systems). The real roots α are such that there exists a $w \in W_r$ sending α to an α_i. We are only concerned with the real roots, but distinguishing between a real root and an imaginary root by inspection is not an easy task. Indeed, any root α is of the form

$$\alpha = \sum_{i=1}^{r} k_i \alpha_i + k_{r+1}\alpha_{r+1} \quad (k_i \in \mathbb{Z} \text{ for } 1 \leq i \leq r),$$

and there's not much here to directly tell us if α is real. Nevertheless, we have an inductive argument to compute real roots α with fixed k_{r+1}. Computing the real roots for fixed k_{r+1} is sufficient, but we would like a closed formula to compute the real roots.

References

[1] J.B. Conrey, D.W. Farmer, J.P. Keating, M.O. Rubinstein and N.C. Snaith: *Integral moments of L-functions*. Proc. London Math. Soc. (3) **91** (2005), no. 1, 33–104.

[2] A. Diaconu and I. Whitehead: *On the third moment of* $L\left(\frac{1}{2}, \chi_d\right)$ *II: the number field case*. J. Eur. Math. Soc. (JEMS), to appear. Available at arXiv:1804.00690.