Whittaker functions and the alcove walk model

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Outline

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- Basic tool in aut. forms and the construction of L-functions.
- Arise as common eigenfunctions in physics.

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Upshot: We view (π, V) as functions on $G(F)$.

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This is the (spherical) Whittaker function.

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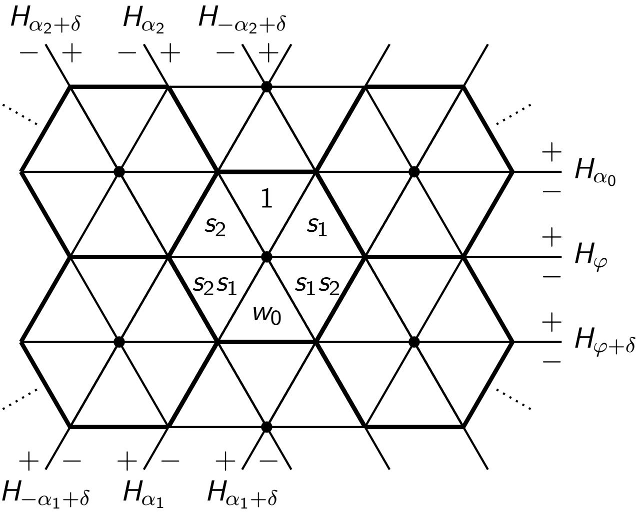

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$$\langle \alpha, \beta \rangle = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}.$$

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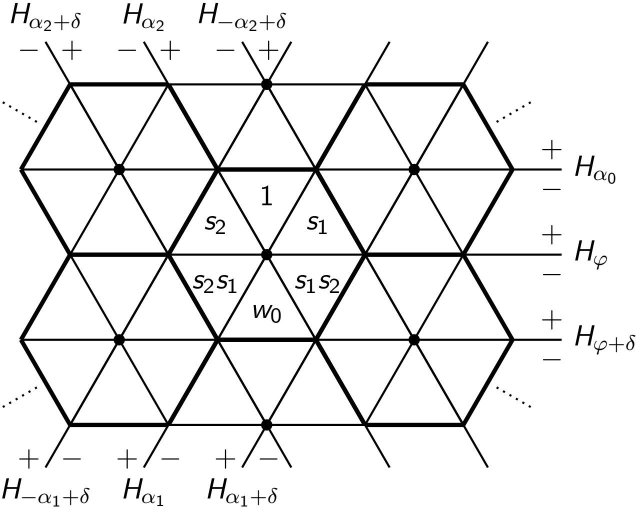

Hyperplanes and Reflections

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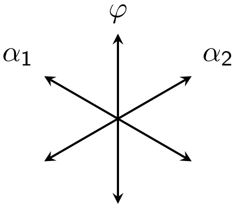

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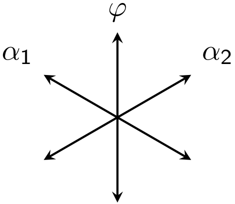

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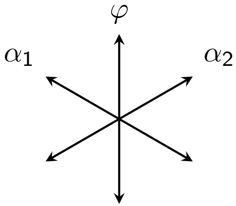

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The highest root is

$$\varphi = \alpha_1 + \alpha_2.$$

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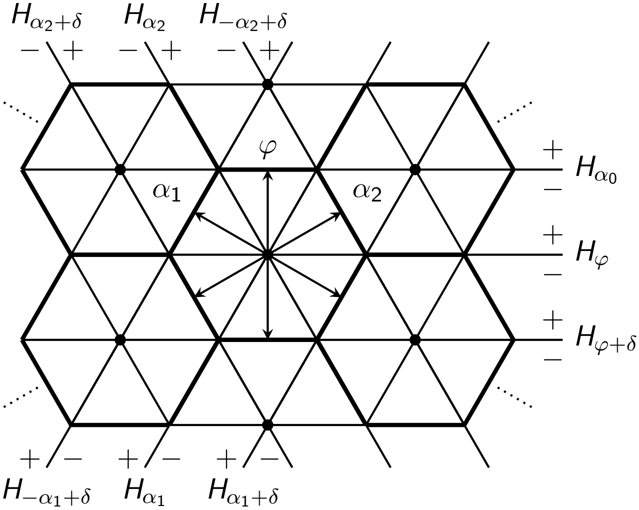

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Also,

$$W_{\text{aff}} \cong W \rtimes Q^\vee$$

under translation by $Q^\vee$.

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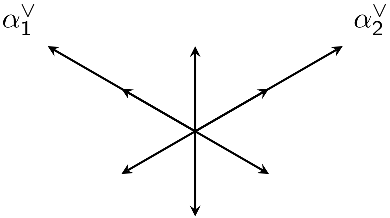

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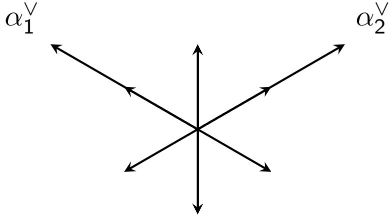

$$Q^\vee = \mathbb{Z}\alpha_1^\vee + \mathbb{Z}\alpha_2^\vee \quad \text{and} \quad \widetilde{S}_3 \cong S_3 \ltimes Q^\vee.$$

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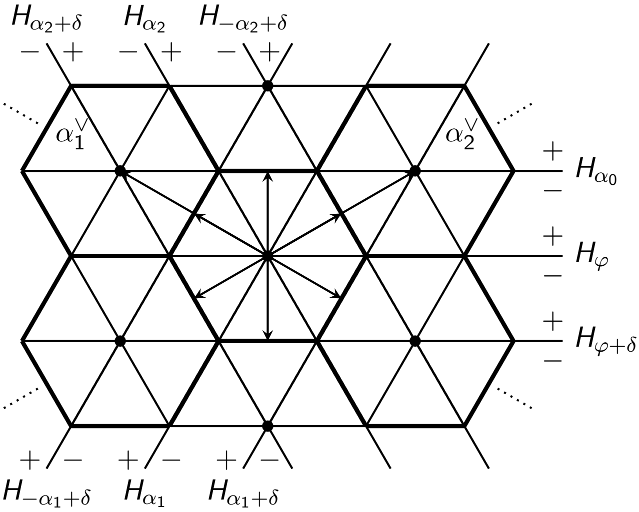

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These facts dictate most of the combinatorics about the walk.

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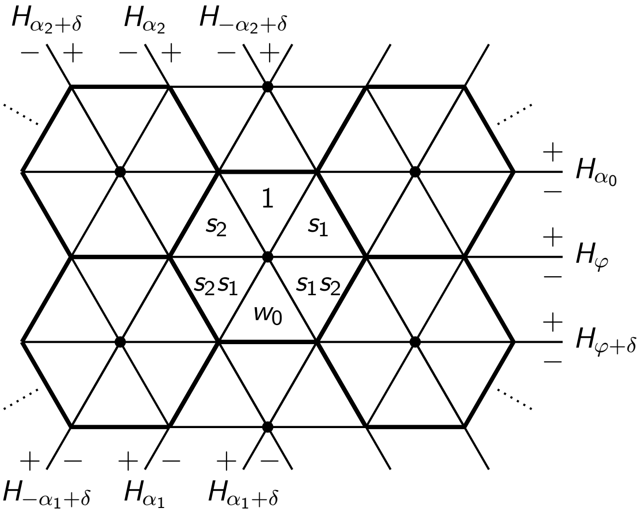

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1. Whittaker Functions
2. The Alcove Walk Model
3. An Introduction to Folding
4. References

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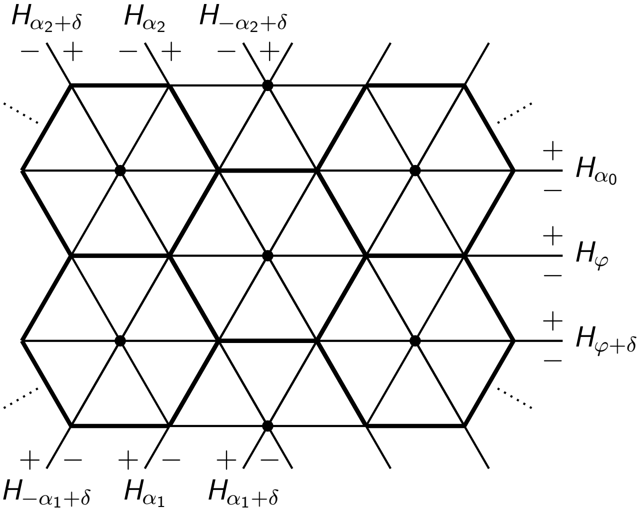

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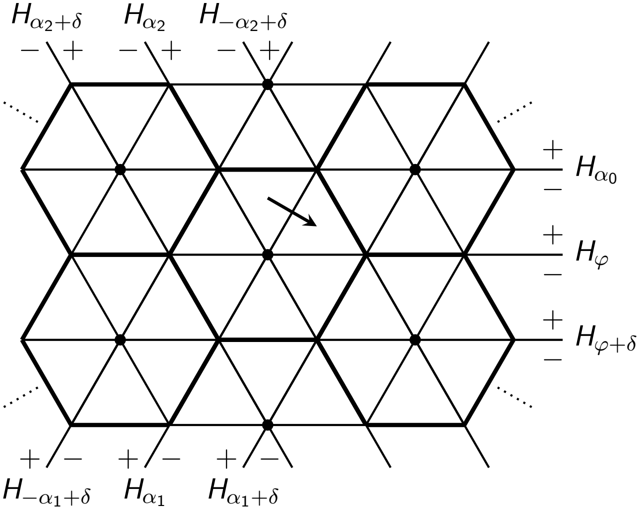

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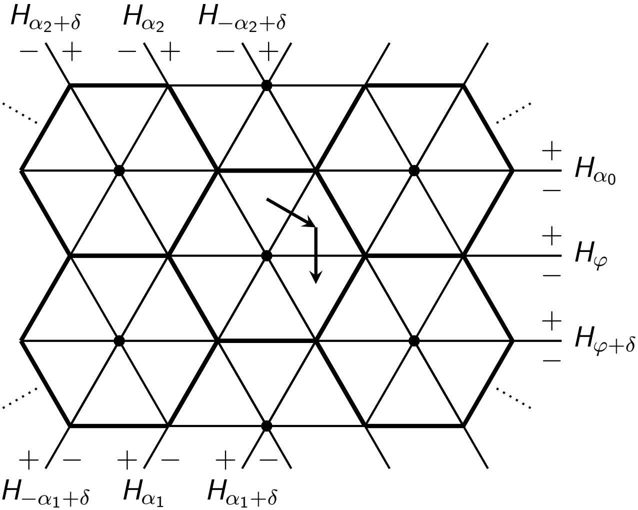

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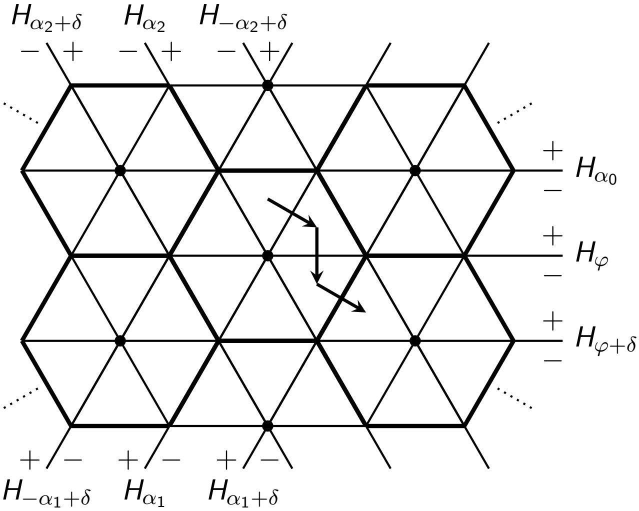

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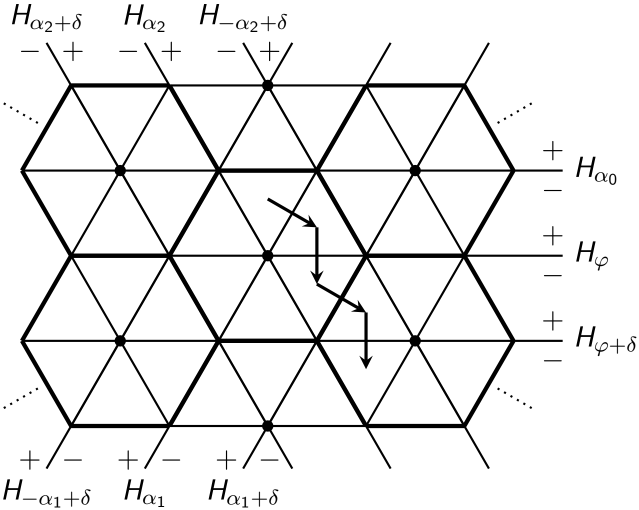

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We label each step of the walk with elements in a finite field $\mathbb{F}_q$.

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$$IwI \longleftrightarrow \left\{ \begin{matrix} \text{walks to } w \text{ with} \\ \text{labels in } \mathbb{F}_q \end{matrix} \right\}.$$

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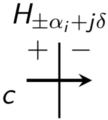
Turns steps of the form



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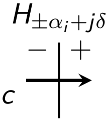

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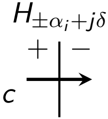

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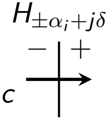

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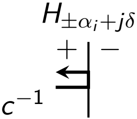

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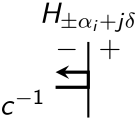

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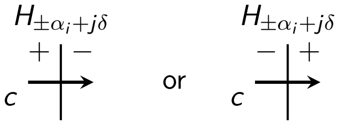

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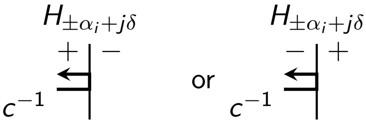

If $c = 0$ we cannot fold and if $c \neq 0$ we must fold!

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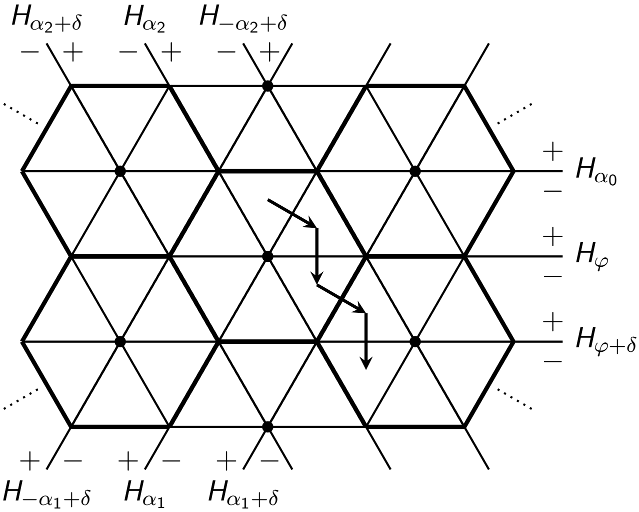

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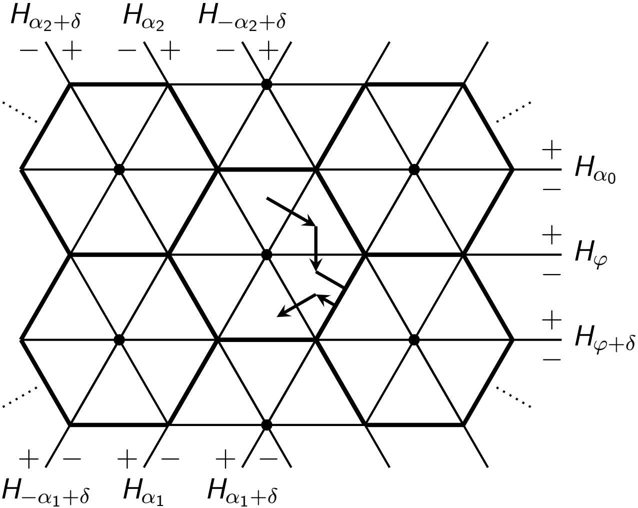
Let $w = s_1 s_2 s_0 s_1$ with labels $(0, 0, c, 0)$. The folded walk is



Folded Walks and Double Cosets

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This last bijection makes our Whittaker functions extremely computable!

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References

We thank the NSF RTG grant supporting this work with grant no. DMS-1745638.

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