Combinatorics of the alcove walk model

Henry Twiss

University of Minnesota

October 2020

Outline

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The Alcove Walk Model for The Plane

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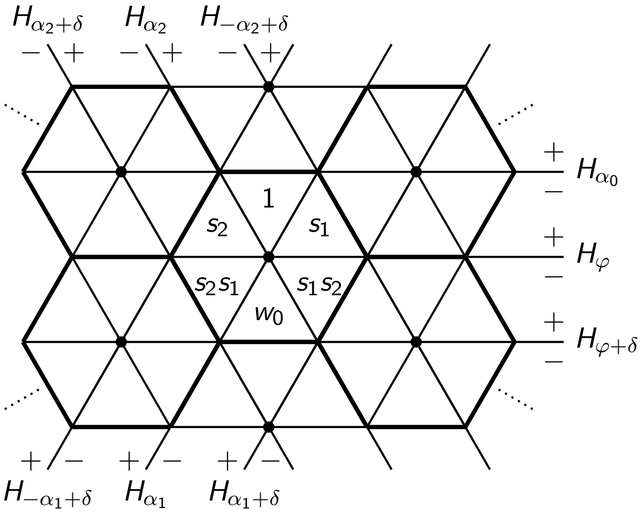

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- The lines $H_{\pm\alpha_i + j\delta}$ and the vectors α_i.
- The signage $+$ and $-$ attached to each line.

The Setting

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$\langle \cdot, \cdot \rangle$ is not an inner product!

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is the finite Weyl group. The longest element of W is

$$w_0 = s_1 s_2 s_1 = s_2 s_1 s_2.$$

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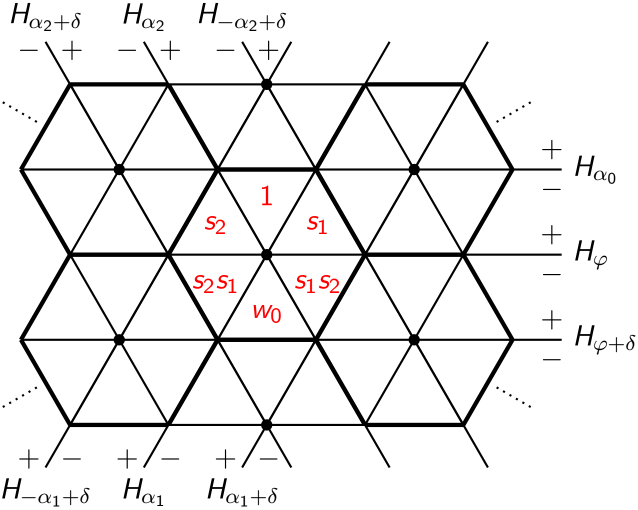

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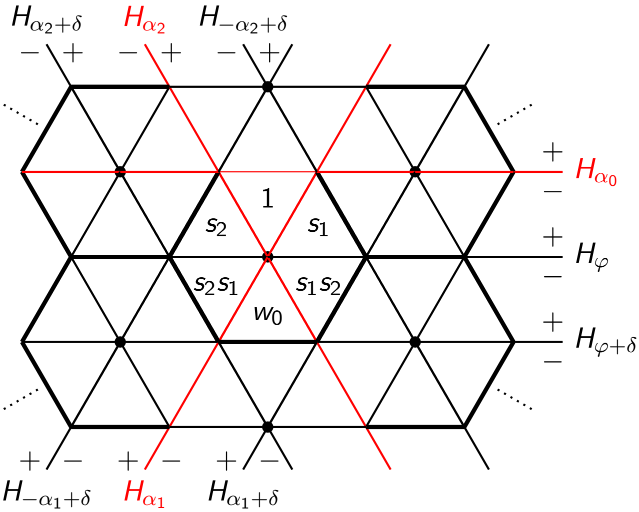

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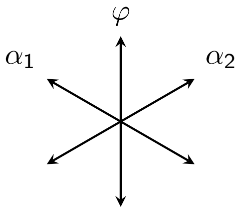

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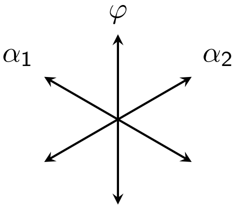

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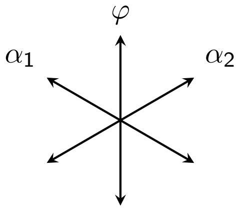

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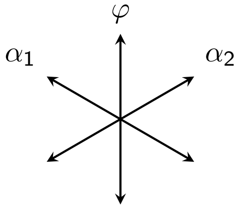

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$$\varphi = \alpha_1 + \alpha_2.$$

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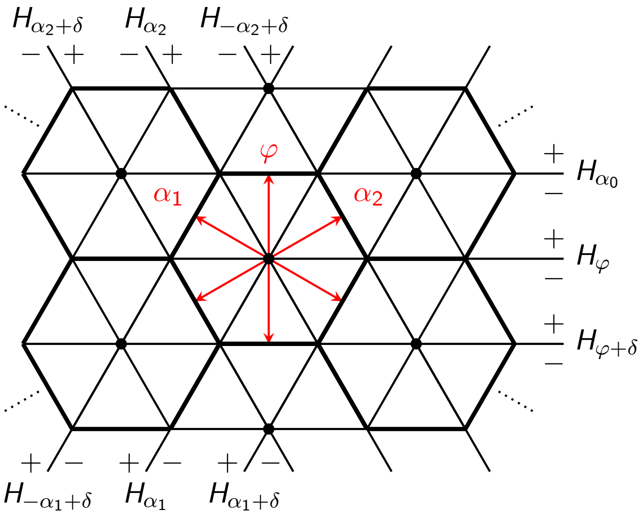

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These facts dictate most of the combinatorics about walks!

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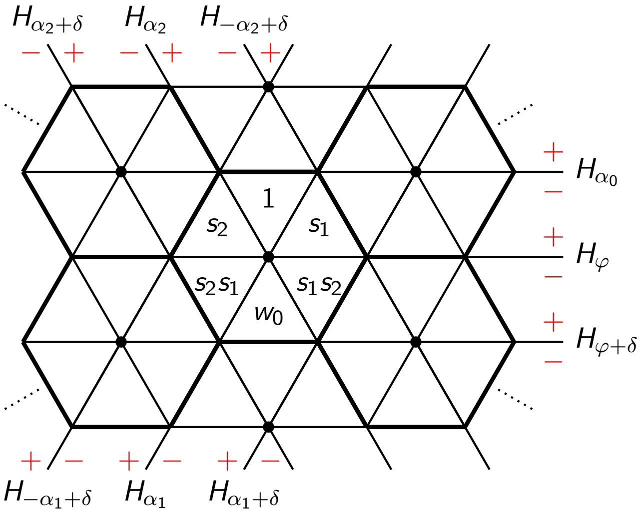

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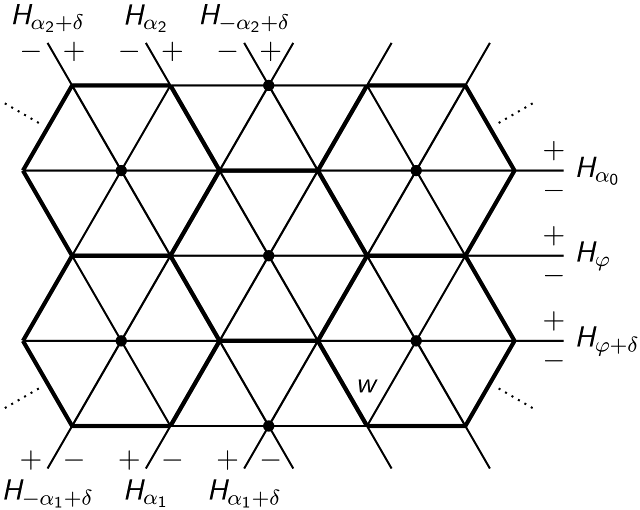
A minimal walk to $w = s_1 s_2 s_0 s_1$ is



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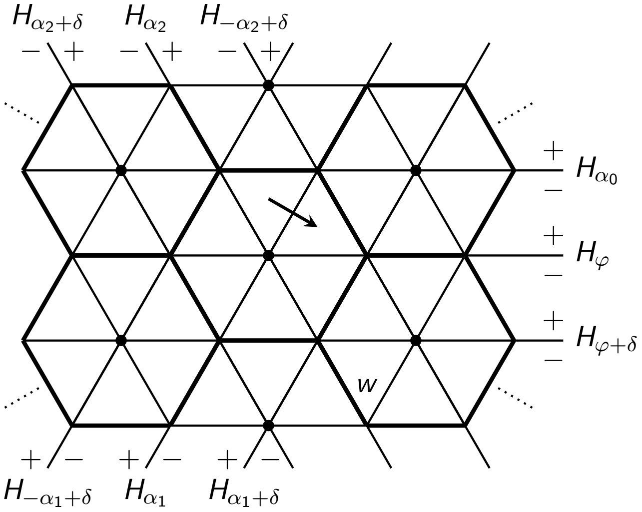

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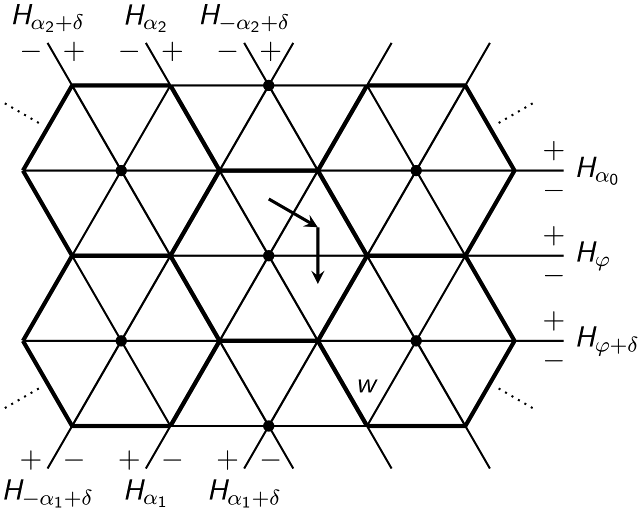

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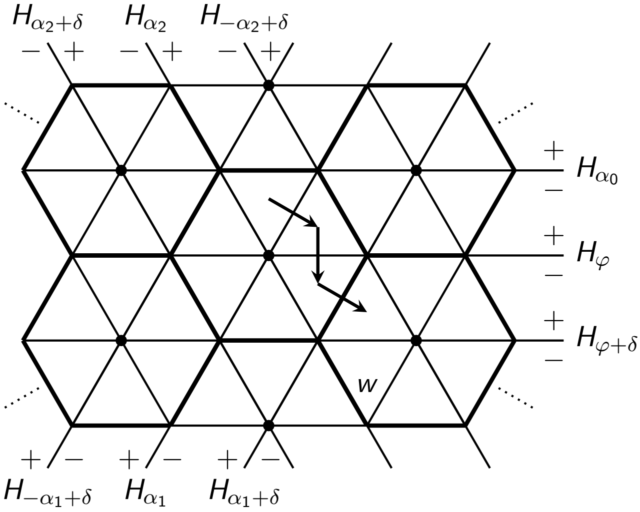

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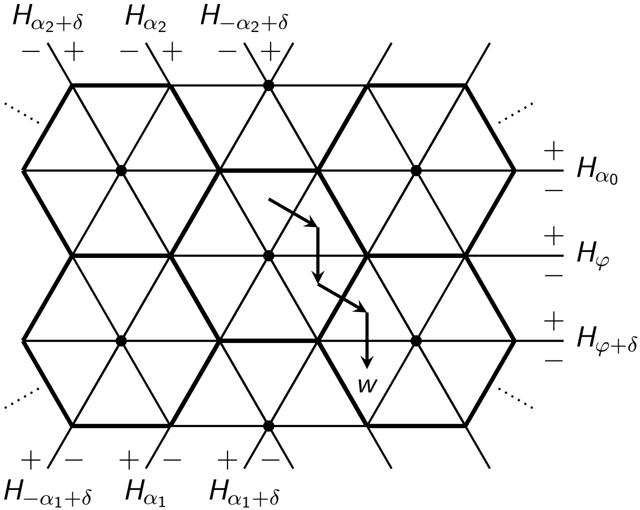

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Folding

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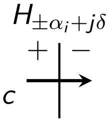
Positive (resp. negative) folding steps of the form



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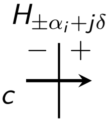

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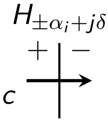

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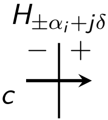

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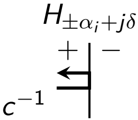

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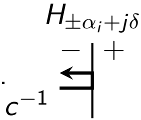

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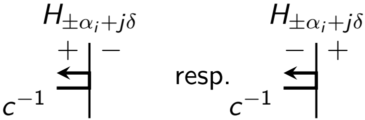

We declare that if $c = 0$ we cannot fold and if $c \neq 0$ we must fold!

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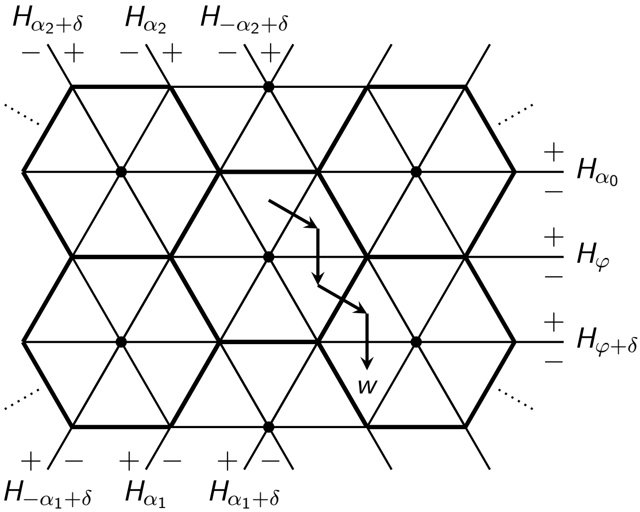

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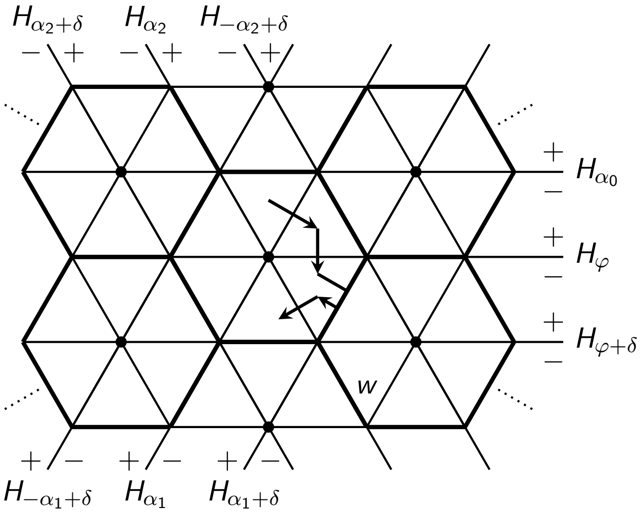

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- What are the maximum numbers of folds possible in any walk?
- Given elements $w_1, w_2 \in W_{\text{aff}}$ does there exist a labeled walk to w_1 that positively folds to w_2?

Question 1

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Theorem 3.1

If a labeled walk positively folds at step k across a hyperplane of type $H_{\pm\alpha_i+j\delta}$, then the folded walk is obtained from w by introducing a folded step at step k and reflecting the tail across $H_{\pm\alpha_i+j\delta}$.

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Theorem 3.1

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This makes it very easy to compute folded walks!

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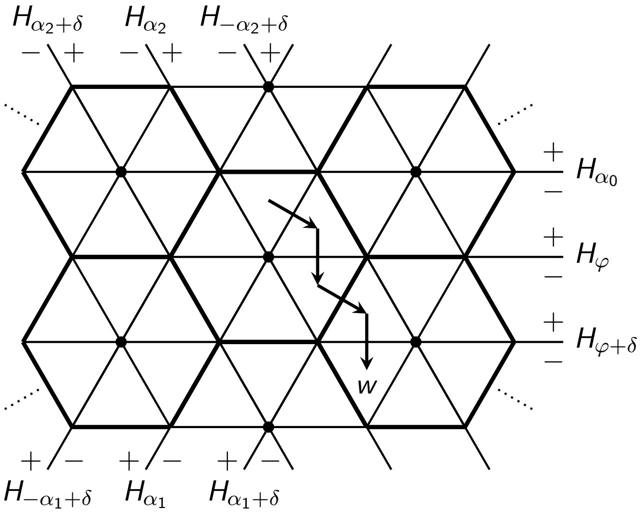

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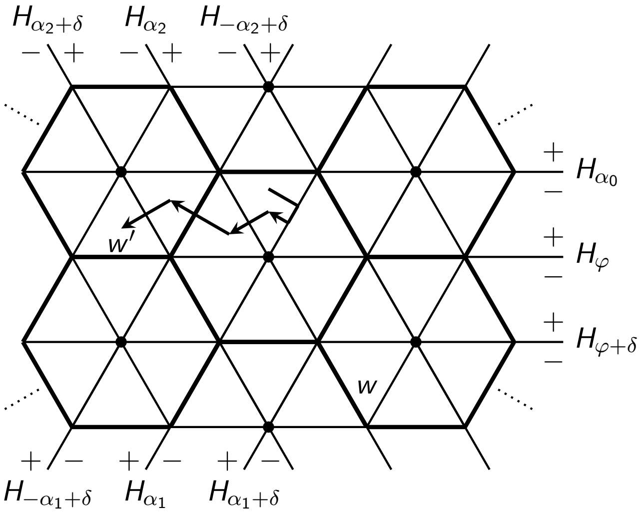

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Replacing a walk with a fold means we **don't** cross a hyperplane at that step so we need to remove the reflection we applied:

$$s_i^2 = 1.$$

Shape Sets

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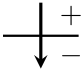
Steps of the form



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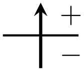

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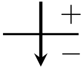

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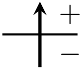

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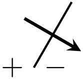

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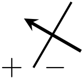

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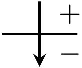

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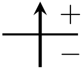

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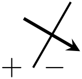

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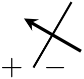

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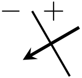

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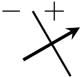

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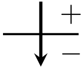

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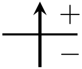

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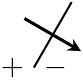

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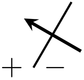
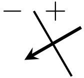


Steps of shape $\mp\alpha_2$ are



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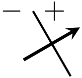

Each walk w determine a shape set S_w.

An Example

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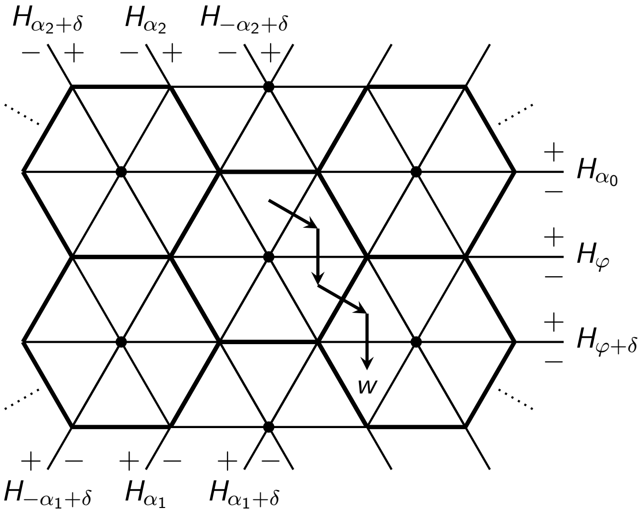
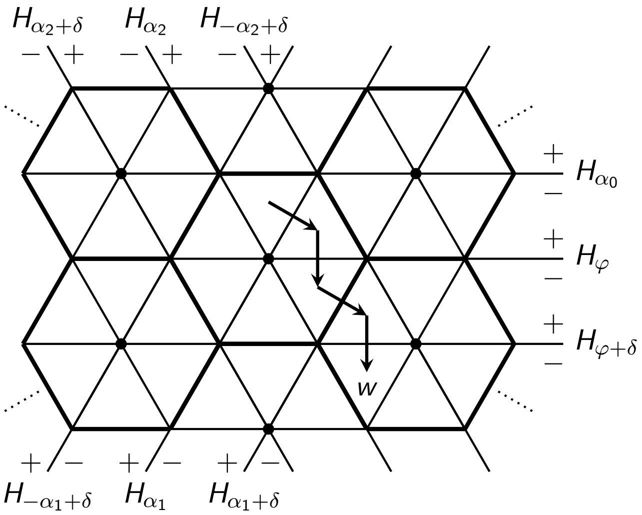

$H_{\alpha_2+\delta}$
H_{α_2}
$H_{-\alpha_2+\delta}$
$-$ $+$
$-$ $+$
$-$ $+$
$+$ H_{α_0} $-$
$+$ H_{φ} $-$
$+$ $H_{\varphi+\delta}$ $-$
w
$+$ $-$
$+$ $-$
$+$ $-$
$H_{-\alpha_1+\delta}$
H_{α_1}
$H_{\alpha_1+\delta}$

An Example



The shape set: $S_w = \{-\varphi, -\alpha_1, \alpha_2\}$.

Question 2

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Theorem 3.2

If w is any walk, the maximum number of positive folds is $\ell(w_0) = 3$.

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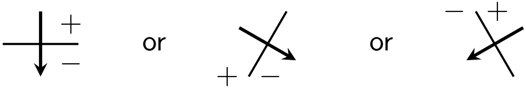

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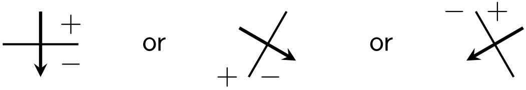

Careful, S_w changes after we fold! It does so in a way that decreases the total amount of $-$ shapes in S_w.

Question 3

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Very very hard!

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Difficulty: Hard to keep track of the data of a walk after we fold multiple times.

Outline

References

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