Motivations

Root Systems from Lie Algebras

Additive Structure

Real Primitive Sets

References

# Root systems attached to moments of quadratic Dirichlet L-functions

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# Outline

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# Dirichlet $L$-Functions and Moments

Dirichlet $L$-function (over $A = \mathbb{F}_q[T]$):

$$L(s, \chi_d) = \sum_{f \text{ monic}}^{\infty} \frac{\chi_d(f)}{|f|^s} \qquad \Re(s) > 1$$

where

$$\chi_d : A/dA \to \mathbb{C} \quad \chi_d(m) = (d/m).$$

Prototypical example:

$$L(s, 1) = \zeta_A(s) = \frac{1}{1 - q^{1-s}}.$$

$r$-th moment at the central value:

$$M_r(D) := \sum_{\substack{d-\text{monic sq. free} \\ \deg(d)=D}} L(\tfrac{1}{2}, \chi_d)^r.$$

# Asymptotics

## Theorem (Diaconu-Twiss, 2020)

Let $D \geq 1$ integers and $r \geq 4$, then under a few conjectures,

$$M_r(D) = \sum_{n=1}^{\infty} Q_n(D,q) q^{\left(\frac{1}{2}+\frac{1}{2n}\right)D} + O_{\epsilon,q,r}\left(q^{D\left(\frac{1}{2}+\epsilon\right)}\right).$$

We want to understand the polynomials $Q_n(D,q)$:

# Asymptotics

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We want to understand the polynomials $Q_n(D,q)$:

$$Q_n(D,q) = n^{-1} \cdot \sum_{\alpha \in \Phi_n} \left\{ \sum_{\substack{a \\ \zeta_a^n = \mathrm{sgn}(a)}} \frac{\Gamma_{w_\alpha(1,a;\zeta_a)} S_\alpha\left(\frac{1}{2}, \zeta_a\right) \zeta_a^D}{2^{l(w_\alpha)}} \right\} q^{\left(\frac{n+1}{2n}\right)D}.$$

# Asymptotics

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$\Phi_n$ is a set of real roots in a root system.

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# Generalized Cartan Matrices

Generalize Cartan matrix $A = (A_{ij})_{i,j=1}^{r+1}$:

1. $A_{ii} = 2$.
2. $A_{ij} = 0$ if and only if $A_{ji} = 0$.
3. $A_{ij} \leq 0$.

The generalized Cartan matrix $A_{r+1}$ of interest:

$$A_{r+1} = \begin{pmatrix} 2 & & & -1 \\ & 2 & & \vdots \\ & & \ddots & -1 \\ -1 & \cdots & -1 & 2 \end{pmatrix}$$

Not Type $A$ in general.

# Cartan Matrices

You can build a unique Lie algebra $\mathfrak{g}(A_{r+1})$ from a generalized Cartan matrix!

1. Consider $\mathfrak{h} := \mathbb{C}^{2n-\text{rank}(A_{r+1})}$.
2. Construct vectors $\alpha_i^\vee$ and dual vectors (simple roots) $\alpha_j$ satisfying $\langle \alpha_i^\vee, \alpha_j \rangle = \alpha_j(\alpha_i^\vee) = (A_{r+1})_{ij}$.
3. Use Chevalley-Serre relations.

There exists a symmetric invariant bilinear form:

$$(- \mid -) = \frac{1}{2}\langle -, - \rangle.$$

# The Kac-Moody Lie Algebra of $A_{r+1}$



Root space decomposition:

$$\mathfrak{g}(A_{r+1}) = \mathfrak{h} \oplus \left( \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \right)$$

Eigenspaces of adjoint rep:

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g}(A) \mid [h, x] = \langle h, \alpha \rangle x \text{ for all } h \in \mathfrak{h}\}$$

where $\alpha \in \Delta$ if and only if $\mathfrak{g}_\alpha \neq 0$.

$\Delta$ is the set of roots of $\mathfrak{g}(A_{r+1})$

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Simple roots attached to $\mathfrak{g}(A_{r+1})$:

$$\Pi = \{\alpha_i\}_{i=1}^{r+1} \subset \mathfrak{h}^*.$$

The root lattice:

$$\Delta \subset Q = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \cdots + \mathbb{Z}\alpha_{r+1},$$
$$\Delta_+ \subset Q_+ = \mathbb{Z}_{\geq 0}\alpha_1 + \mathbb{Z}_{\geq 0}\alpha_2 + \cdots + \mathbb{Z}_{\geq 0}\alpha_{r+1}.$$

## Example

$$\alpha = 3\alpha_{i_1} + 2\alpha_{i_2} + 2\alpha_{i_3} + \alpha_{i_4} + \alpha_{i_5} + 6\alpha_{r+1}$$
$$\alpha \longleftrightarrow (3, 2, 2, 1, 1, 6)$$

# Weyl Groups

Simple reflections:

$$w_i(\lambda) = w_{\alpha_i}(\lambda) := \lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i.$$

The Weyl group $W_{r+1}$:

$$W_{r+1} = \langle w_i \mid 1 \leq i \leq r+1 \rangle.$$

Facts:

1. $W_{r+1}\Delta = \Delta$.
2. $w_i^2 = 1$, $w_i(\alpha_i) = -\alpha_i$, $w_i w_j = w_j w_i$ for $1 \leq i, j \leq r$.

Action of simple reflections on $\alpha = \sum_{i=1}^{r+1} k_i \alpha_i$:

$$w_i : k_i \mapsto k_{r+1} - k_i \qquad (1 \leq i \leq r)$$

$$w_{r+1} : k_{r+1} \mapsto \sum_{i=1}^{r} k_i - k_{r+1} \qquad i = r+1.$$

# Real and Imaginary Roots

Real roots:

$$\Delta^{\mathsf{re}} = W_{r+1}\Pi.$$

Imaginary roots:

$$\Delta^{\mathsf{im}} = \Delta - \Delta^{\mathsf{re}}.$$

Imaginary roots do not exist in the finite dimensional setting!

## Example

$$\delta = \alpha_{i_1} + \alpha_{i_2} + \alpha_{i_3} + \alpha_{i_4} + 2\alpha_{r+1}$$
$$\delta \longleftrightarrow (1,1,1,1,2)$$

# A Real Root



$$(3, 2, 2, 1, 1, 6)$$

# A Real Root



$$(3,2,2,1,1,6)$$
$$\downarrow$$
$$(3,2,2,1,1,\color{red}{3})$$

$$\overset{\color{red}{w_{r+1}}}{6 \mapsto 3+2+2+1+1-6}$$

# A Real Root

$$(3, 2, 2, 1, 1, 6)$$
$$\downarrow$$
$$(3, 2, 2, 1, 1, 3)$$
$$\downarrow$$
$$(0, 1, 1, 1, 1, 3)$$

$$w_1 w_2 w_3 w_{r+1}$$

$$3 \mapsto 3 - 3, \quad 2 \mapsto 3 - 2, \quad 2 \mapsto 3 - 2$$

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$$(3,2,2,1,1,6)$$
$$\downarrow$$
$$(3,2,2,1,1,3)$$
$$\downarrow$$
$$(0,1,1,1,1,3)$$
$$\downarrow$$
$$(0,1,1,1,1,\textcolor{red}{1})$$

$$\textcolor{red}{w_{r+1}}w_1w_2w_3w_{r+1}$$
$$3 \mapsto 0+1+1+1+1-3$$

# A Real Root

$$(3,2,2,1,1,6)$$
$$\downarrow$$
$$(3,2,2,1,1,3)$$
$$\downarrow$$
$$(0,1,1,1,1,3)$$
$$\downarrow$$
$$(0,1,1,1,1,1)$$
$$\downarrow$$
$$(0,0,0,0,0,1)$$

$$w_2 w_3 w_4 w_5 w_{r+1} w_1 w_2 w_3 w_{r+1}$$

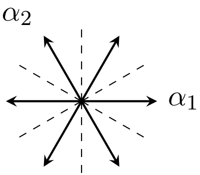
$$1 \mapsto 1-1, \quad 1 \mapsto 1-1, \quad 1 \mapsto 1-1, \quad 1 \mapsto 1-1$$

# Types

$\mathfrak{g}(A_{r+1})$ is exactly one of the following types:

1. Finite: $r = 1, 2, 3$ (good)

$$\Delta^{\text{im}} = \varnothing \quad \text{and} \quad (\alpha \mid \alpha) = 1 \iff \alpha \in \Delta^{\text{re}}$$



# Types

$\mathfrak{g}(A_{r+1})$ is exactly one of the following types:

1. Finite: $r = 1, 2, 3$ (good)
2. Affine: $r = 4$ (good)

$$\Delta^{\text{im}} = (\mathbb{Z} - \{0\})\delta \quad \text{and} \quad (\alpha \mid \alpha) = 1 \iff \alpha \in \Delta^{\text{re}}$$

# Types

$\mathfrak{g}(A_{r+1})$ is exactly one of the following types:

1. Finite: $r = 1, 2, 3$ (good)
2. Affine: $r = 4$ (good)
3. Hyperbolic: $r = 5$ (less good)

$$(\alpha \mid \alpha) = 1 \iff \alpha \in \Delta^{\text{re}}$$

# Types

$\mathfrak{g}(A_{r+1})$ is exactly one of the following types:

1. Finite: $r = 1, 2, 3$ (good)
2. Affine: $r = 4$ (good)
3. Hyperbolic: $r = 5$ (less good)
4. Indefinite: $r > 5$ (not good)

$$\alpha \in \Delta^{\text{re}} \implies (\alpha \mid \alpha) = 1$$

We may assume $r > 5$ from now on.

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# Root Strings

For $\beta \in \Delta^{\text{re}}$, $\alpha \in \Delta$:

$$S(\beta, \alpha) = \{\alpha + t\beta \mid -u \leq t \leq v\} \subset \Delta.$$

Moreover, $u - v = \langle \beta^{\vee}, \alpha \rangle$.

**Theorem (Twiss, 2021)**

For $\mathfrak{g}(A_{r+1})$, the string $S(\beta, \alpha)$ is one of the two forms:

$$\bullet \quad \circ \quad \cdots \quad \circ \quad \bullet$$

$$\circ \quad \cdots \quad \circ$$

with $\bullet$ real and $\circ$ imaginary.

For which $\gamma \in \Delta^{\text{im}}$ is $S(\beta, \gamma) = S(\beta, \alpha)$ for $\alpha \in \Delta^{\text{re}}$?

# Strictly Imaginary Roots

Strictly imaginary roots:

$$\Delta^{\text{sim}} = \{\gamma \in \Delta^{\text{im}} \mid \gamma + \beta \text{ or } \gamma - \beta \text{ is a root for all } \beta \in \Delta^{\text{re}}\}$$

$\Delta_{+}^{\text{sim}}$ is a semigroup under addition!

**Theorem (Twiss, 2021)**

For $\mathfrak{g}(A_{r+1})$, we have $\Delta^{\text{im}} = \Delta^{\text{sim}}$.

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# The Sets $\Phi_n$

$$\Phi_n = \{\alpha \in \Delta^{\text{re}} \mid \alpha \text{ primitive and } \alpha_{r+1}\text{-coefficient is } n\}.$$

Primitivity means the $\alpha_i$-coefficients are uniformly bounded by $n/2$ for all $1 \leq i \leq r$.

# Some Examples



$$\Phi_1 = \varnothing,$$

$$\Phi_2 = \{1112\},$$

$$\Phi_3 = \{11113\},$$

$$\Phi_4 = \left\{\begin{matrix}111114\\22214\end{matrix}\right\},$$

$$\Phi_5 = \left\{\begin{matrix}1111115\\221115\\22225\end{matrix}\right\},$$

$$\Phi_6 = \left\{\begin{matrix}11111116\\322116\\33326\end{matrix}\right\},$$

$$\Phi_7 = \left\{\begin{matrix}111111117\\2221117\\3311117\\322217\\333117\\33337\end{matrix}\right\},$$

$$\Phi_8 = \left\{\begin{matrix}1111111118\\3321118\\322228\\442218\\44438\end{matrix}\right\}.$$

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# Some Examples

$$\Phi_1 = \varnothing,$$

$$\Phi_2 = \{1112\},$$

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$$\Phi_4 = \begin{Bmatrix} 111114 \\ 22214 \end{Bmatrix},$$

$$\Phi_5 = \begin{Bmatrix} 1111115 \\ 221115 \\ 22225 \end{Bmatrix},$$

$$\Phi_6 = \begin{Bmatrix} 11111116 \\ 322116 \\ 33326 \end{Bmatrix},$$

$$\Phi_7 = \begin{Bmatrix} 111111117 \\ 2221117 \\ 3311117 \\ 322217 \\ 333117 \\ 33337 \end{Bmatrix},$$

$$\Phi_8 = \begin{Bmatrix} 1111111118 \\ 3321118 \\ 322228 \\ 442218 \\ 44438 \end{Bmatrix}.$$

# Incomparability Property

Every root in $\Phi_n$ is "squeezed" between the red root and blue root.

## Theorem (Twiss, 2021)

Distinct pairs $\alpha, \beta \in \Phi_n$ are incomparable ($\alpha - \beta \notin Q_+ \cup Q_-$).

# Bad Sets

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$$B_1 = \varnothing,$$
$$B_2 = \varnothing,$$
$$B_3 = \varnothing,$$
$$B_4 = \varnothing,$$
$$B_5 = \varnothing,$$
$$B_6 = \varnothing,$$
$$B_7 = \{31111117\},$$
$$B_8 = \{42111118\},$$
$$B_9 = \left\{\begin{matrix} 321111119 \\ 42211119 \end{matrix}\right\}.$$

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- A. Diaconu, H. Twiss: Secondary terms in the asymptotics of moments of $L$-functions. Preprint, 2020; arXiv:2008.13297.
- R.V. Moody: Root systems of hyperbolic type. Adv. in Math. 33 (1979), no. 2, 144–160.
- Y. Billig, A. Pianzola: Root strings with two consecutive real roots, Tohoku Math. J. (2) Volume 47, Number 3 (1995),
- V. Kac: Infinite Dimensional Lie Algebras, Third Edition, University of Cambridge (1990).